Institute for Advancing Intelligence (IAI), TCG CREST
Mid-Semestral Examination
Ph.D Program Session: 2021–2022
Discrete Mathematics

Date: 07.10.2020 Marks: 60 Time: 4 Hours

Answer as much as you can. Total marks is 75 and the maximum you can score is 60. This is a open book, open note examination. Copying or Discussion among yourselves is strictly prohibited.

1. (a) Justify whether the following argument is correct or incorrect:

   Tamojit likes all Indian cricketers. Tamojit likes Virat Kohli. Therefore, Virat Kohli is an Indian cricketer.

   (b) Use quantifiers to express the following statement:

   There does not exist any man who has visited all the countries in the world.

   (c) On a table, a row of 50 coins of various denominations, is placed. Shreya picks a coin from one of the ends and puts it in her pocket; then Arkapal chooses a coin from one of the ends and the alternation continues until Arkapal pockets the last coin. Devise a strategy for Shreya so that she wins at least as much money as Arkapal.

   (d) Find fallacy in the following inductive proof.

   Claim: Prove that, for $\frac{d}{dx}(x^n) = 0,\ \forall n \geq 0$.

   Proof. We prove it via induction as follows:

   Base case: $n = 0$: $\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = 0$

   Induction hypothesis: Assume that $\frac{d}{dx}(x^m) = 0$, for all $m \leq n$.

   Inductive Step: Here we show that the statement is true for $n+1$ as follows:

$$\begin{aligned} \frac{d}{dx}(x^{n+1}) &= \frac{d}{dx}(x^n.x) \\ &= x.\frac{d}{dx}(x^n) + x^n.\frac{d}{dx}(x) = x.0 + x^n.0 = 0 \quad \square \end{aligned}$$

[3+4+5+3=15]

2. (a) Suppose that Hilbert's Grand Hotel is fully occupied on the day the hotel expands to a second building which also contains a countably infinite number of rooms. Show that the current guests can be spread out to fill every room of the two buildings of the hotel.

   (b) A line of n airline passengers is waiting to board a plane. They each hold a ticket to one of the n seats on that flight. (For convenience, lets say that the i th passenger in line has a ticket for the seat number i.) Unfortunately, the first person in line is crazy, and will ignore the seat number on their ticket, picking a random seat to occupy. All of the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random. Show that the number of sitting arrangements for which the last person sits on the first seat and the last person sits on the last seat are same (without explicitly counting the number).

   (c) Jayanta is given a circle with a set of n points lying on the circumference of the circle. Suppose he draws lines connecting every point to every other point. Let I be the set of all points of intersection of the lines in the interior of the circle. Can you help Jayanta to find the cardinality of the set I. Assume that, no three lines intersect in a single point inside the circle.

[4+7+4=15]

3. (a) Saikat is about to order a non-veg pizza at "Trimino's". He first needs to specify the crust (hand tossed/ thin crust/ fresh pan pizza). Then he can choose any two type of meat from chicken, turkey, lamb and pork. Finally he can add any combination of extras. The extras offered are cheese, paneer, corn, red pepper, black olive, mushroom, and jalapeno. How many different pizzas can Saikat order?

   (b) Suppose 10 of you have appeared in this Discrete Mathematics examination. Prove that one can always find two disjoint group of students, whose marks-sum is equal. Remember that this is a 60 marks question. For example, if the marks of the students are: $25, 29, 31, 33, 35, 37, 50, 54, 59, 60$, then you have one group with marks $33, 37, 50$, and another group with marks $25, 29, 31, 35$, both of whose sum is 120.

   (c) Inspector Sagnik has to put some criminals in cells of the prison.

The criminals are notorious and can beat one another to death. If any criminal dies inside the cell, then the inspector will lose his job. In this scenario, Sagnik thought of putting each criminal in a cell. But, his boss wants it to be done using the minimum number of cells. The only saving grace for the inspector is that the criminals fight according to the following pattern: (i) a criminal does not beat himself, (ii) if a criminal C_1 does not beat a criminal C_2, and criminal C_2 does not beat criminal C_3, then criminal C_1 does not beat criminal C_3. Help Sagnik by solving this problem efficiently. [Note: It is possible that criminal C_1 does not beat criminal C_2, but criminal C_2 does beat criminal C_1]

[4+7+4=15]

4. (a) A directed graph $G = (V, E)$ is called "*one-directional*" if for all distinct vertices v_i and v_j the following holds:

$$\text{If } (v_i, v_j) \in E, \text{ then } (v_j, v_i) \notin E.$$

Similarly, a directed graph $G = (V, E)$ is called "*bi-directional*" if for all distinct vertices v_i and v_j the following holds:

$$\text{If } (v_i, v_j) \in E, \text{ then } (v_j, v_i) \in E.$$

Count the number of n-vertices (i) *one-directional*, (ii) *bi-directional*, and (iii) both *one-directional* and *bi-directional* di-graphs.

(b) A graph G is called *Forced Eulerian* from a vertex v if whenever we start from a vertex v and traverse the graph without repeating edges, we end up with an Eulerian trail. An example is given in Figure 1. (i) Prove that an Eulerian graph G is forced Eulerian

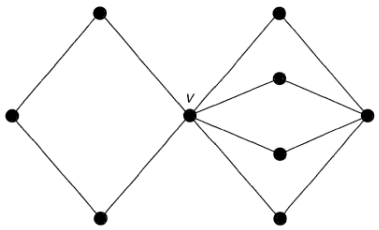
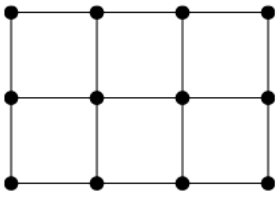


Figure 1: An Example of a Forced Eulerian graph from vertex v.

from a vertex v if and only if every cycle of G contains v. (ii) What can you say about the structure of a graph which is forced Eulerian for all its vertices? **[6+9=15]**

5. (a) Consider a graph G_m that is a grid of $3m$ vertices arranged in 3 rows and m columns such that each vertex has an edge to the vertices to the left, above, to the right, and below it, if they exist. For example the graph G_4 is given below. Find the necessary and



sufficient conditions for G_m to have (i) an eulerian cycle, (ii) a hamiltonian circuit.

(b) Consider a *special knight* that can move three squares vertically followed by one square horizontally, or three squares horizontally followed by one square vertically. For vertical movement, both up and down and for horizontal movement, both left and right directions are allowed. Find the values of m and n, for which you can obtain a *special knight* tour (not necessarily closed) in an $m \times n$ board?

(c) I have asked Shital and Sangita to draw a graph with degree-sequence $(3, 3, 3, 3, 3, 3, 3, 3)$. Is it necessary that the graphs they draw are isomorphic? Justify your answer. **[5+5+5=15]**