A Brief Introduction to the Zero Knowledge Proof

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An Example

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

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P vs NP Problem

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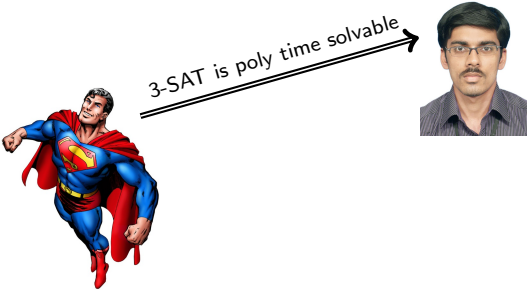
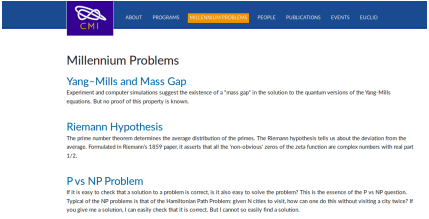

3-SAT is poly time solvable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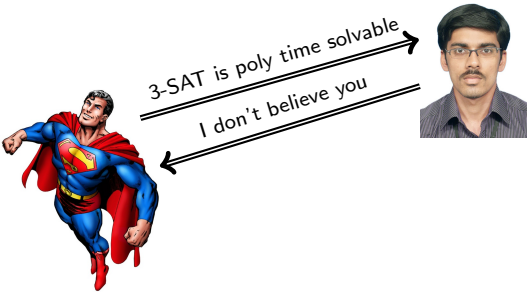
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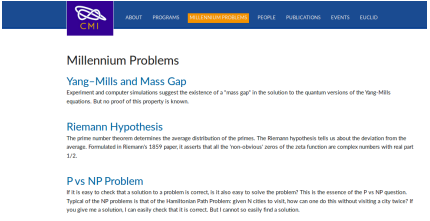
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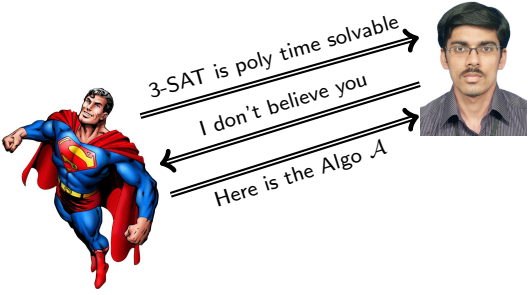
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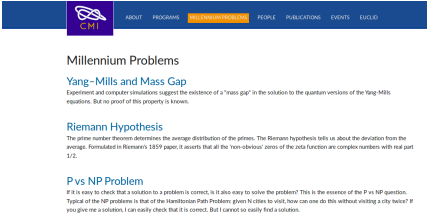
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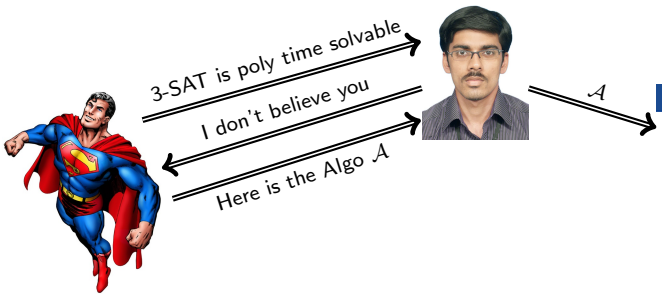
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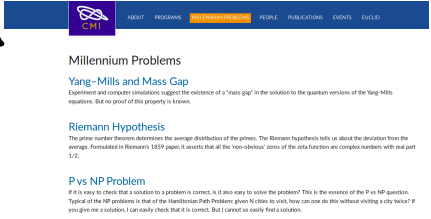
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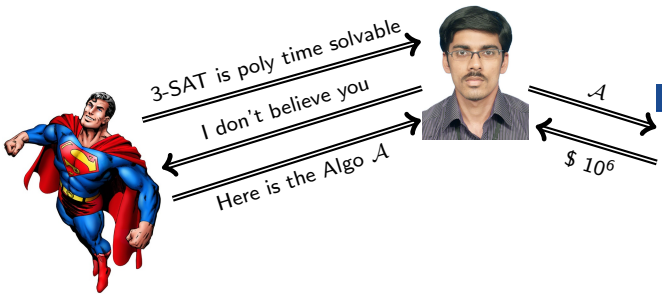
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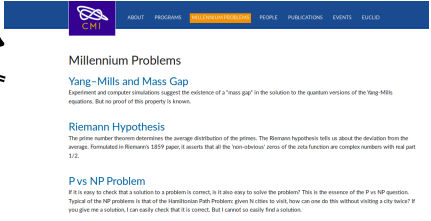
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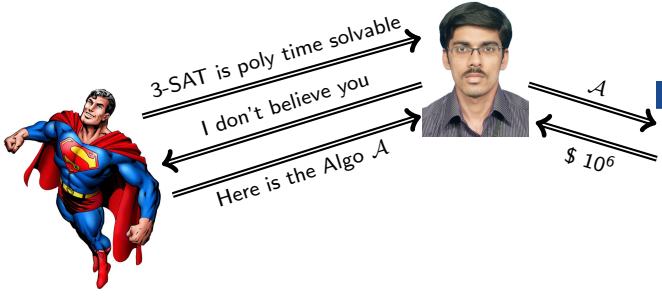
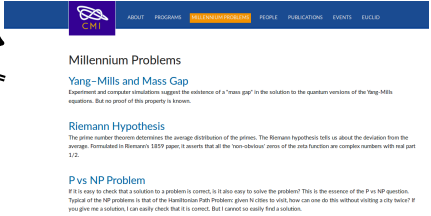
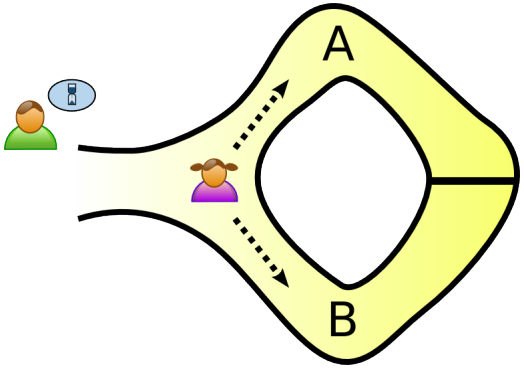




Can Superman convince me the validity of his statement without revealing the proof ?

Motivation: The Story of Alibaba Cave



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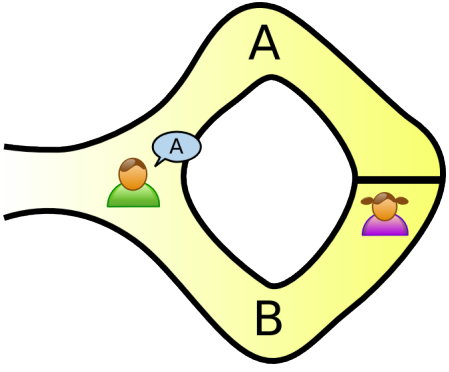

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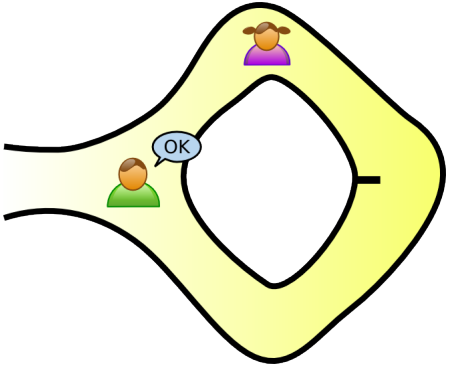

Breakthrough Result of GMR: Zero Knowledge Proof

- Conceived in 1985 by Shafi Goldwasser, Silvio Micali, and Charles Rackoff.
- Their paper "*The Knowledge Complexity of Interactive Proof-Systems*" got accepted in **SIAM'85**.
- Received **Godel Prize** in 1993 for advances in Theoretical Computer Science.

I. Proofs and Proof System

Notion of Proof

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However, a proof makes no sense unless it can be efficiently verified / the verification process is simple.

- SAT has a simple verification procedure ($\in$ NP)
- $\overline{\text{SAT}}$ has no simple verification procedure ($\in$ co-NP)

Types of Proofs

Classical Proof

- The proof is fixed and written somewhere which is either
  - Self-evident, or
  - Derived from self-evident rules.
- These proofs are static in nature.
- Examples are mathematical proofs, proofs for NP-Languages.

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Interactive Proof (IP)

- Proof involves exchanges of information in multiple rounds between prover and verifier
- Truth is established when the verifier accepts the hypothesis
- Proof is dynamic in nature.

A Formal Notion of Proof System

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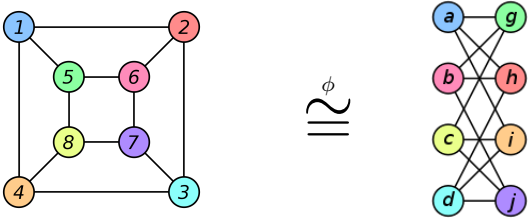
Valid Proof System

- Efficient- Verification of the proof should be simple (run time should be polynomial of the size of the proof)
- Completeness - True statement must have a proof ($\forall x \in \mathcal{L}, \exists \pi$ s.t. $V(x, \pi) = \top$)
- Soundness - False statement does not have any proof ($\forall x \notin \mathcal{L}, \forall \pi$ s.t. $V(x, \pi) = \bot$)

NP- An Example of a Classical Valid Proof System

Graph Isomorphism



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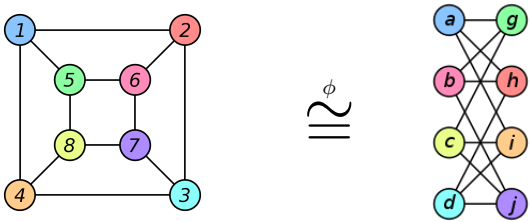

Isomorphism

$$\{\phi(1) = a, \phi(2) = h, \phi(3) = d, \phi(4) = i, \phi(5) = g, \phi(6) = b, \phi(7) = j, \phi(8) = c\}$$

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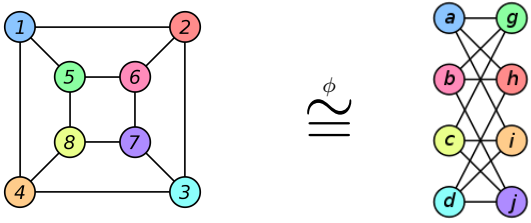

- Prover (P) sends ϕ as the proof π to the verifier (V)
- V verifies π is a valid permutation. Run time is poly.
- Proof is complete and sound.

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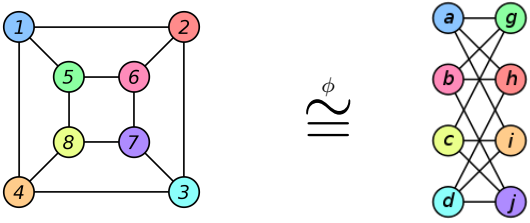

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- NP is the set of languages with classical valid proof system
- No interaction between P and V

Example of a language not in NP

Can we have a language which cannot be proved in classical proof system ?

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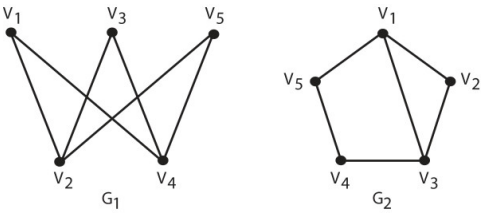
Graph Non-Isomorphism (GNI)



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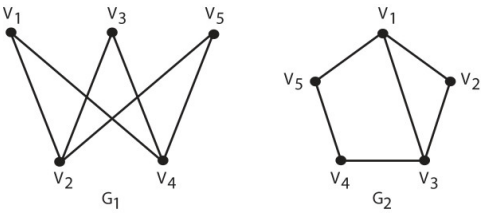

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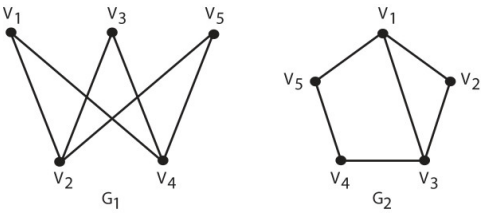

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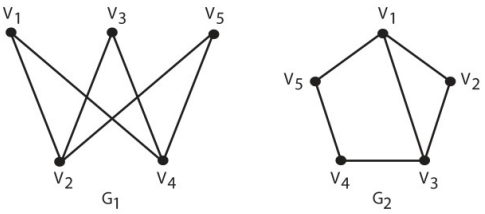

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In fact, we do not know GNI $\in$ NP

Interactive Proof System (IP)

Interactive Protocol

- P wants to prove $x \in \mathcal{L}$
- V is probabilistic in nature – it tosses coins and hence errs with some probability – (Randomness)
- Instead of reading the proof, V $\leftrightarrow$ P to establish the validity of the assertion – (Interactive)
- V finally accepts the proof with some probability.

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- Efficient- Verification of the proof should be simple – (V is a ppt algorithm)
- Completeness - $\forall x \in \mathcal{L}$, Pr[V accepts in (P,V)$(x)$] $= 1$
- Soundness - $\forall x \notin \mathcal{L}, \forall$P*, Pr[V accepts in (P*, V)(x)] $\leq 1/2$

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IP System is said to be "efficient" if P is given the witness

Distinguishing Problem – An Example of IP

P

V

I can distinguish the taste of Pepsi from Coke

Distinguishing Problem – An Example of IP

P

V

I can distinguish the taste of Pepsi from Coke

I don't believe you

Distinguishing Problem – An Example of IP

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How does P prove her claim to V ?

Distinguishing Problem – An Example of IP

P

V

Tell me is it Coke / Pepsi?

$b = 0$, pour pepsi
$b = 1$, pour coke

Distinguishing Problem – An Example of IP

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V

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$b' \in \{0, 1\}$

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Distinguishing Problem – An Example of IP

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V

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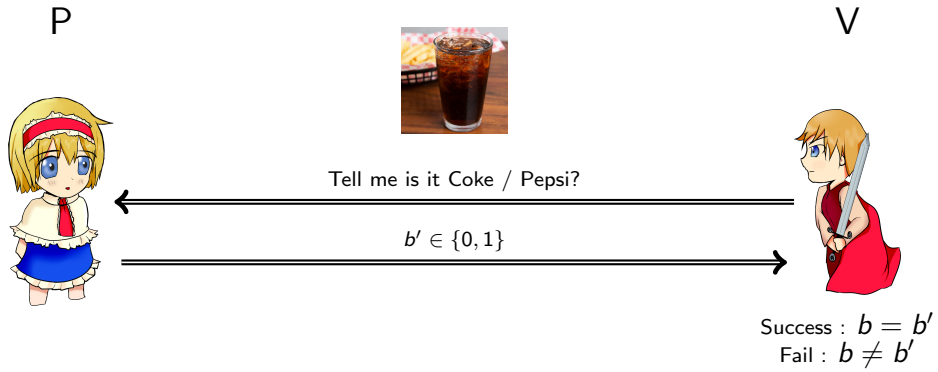
Success : $b = b'$
Fail : $b \neq b'$

Distinguishing Problem – An Example of IP



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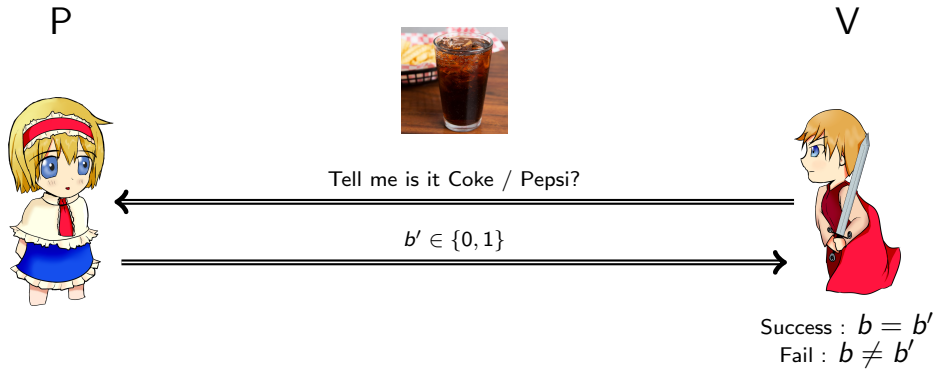

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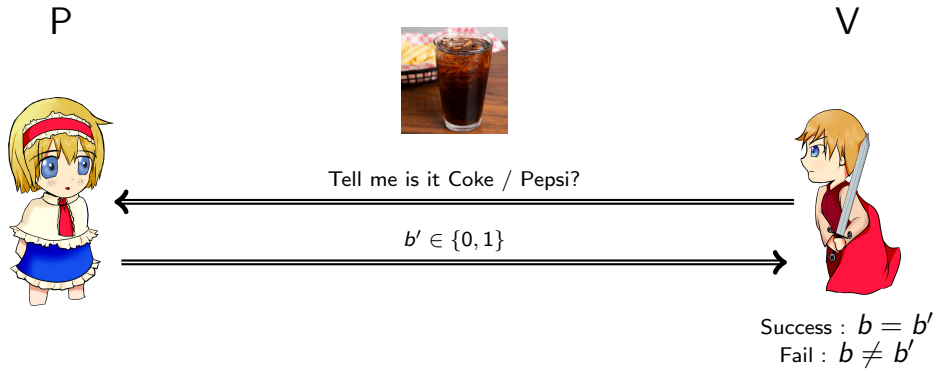

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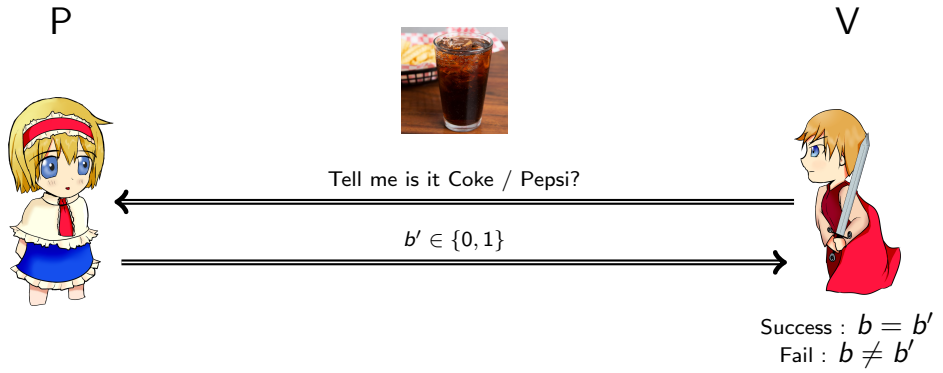
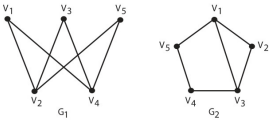


- If P really knows the difference then it always succeeds – **(Complete)**
- If P does not know, then it fails with probability (?) $1/2$
- Repeat the expermient afresh and continues for t times – **Soundness error** 2^{-t}

GNI has an Interactive Proof

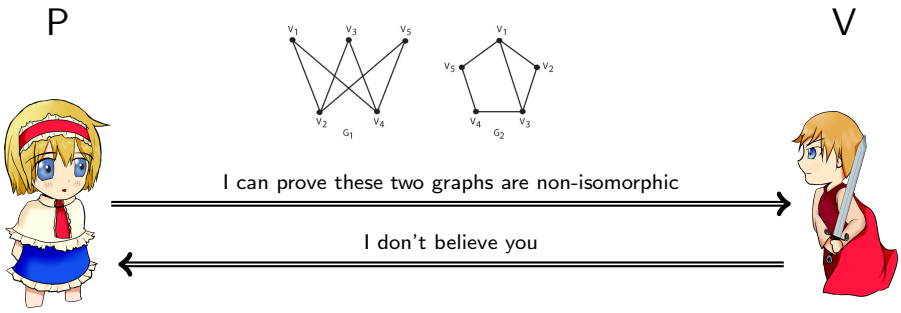
P



V

I can prove these two graphs are non-isomorphic

GNI has an Interactive Proof



GNI has an Interactive Proof

P

V

$\phi(G_b) : \phi \leftarrow \mathrm{Perm}(V)$

Tell me is it isomorphic to G_1/G_2?

$b = 0$, select G_1
$b = 1$, select G_2

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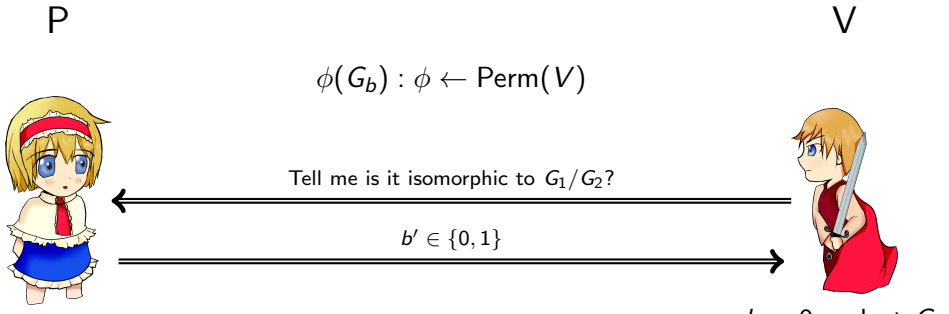

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V

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$b' \in \{0, 1\}$

Success : $b = b'$
Fail : $b \neq b'$

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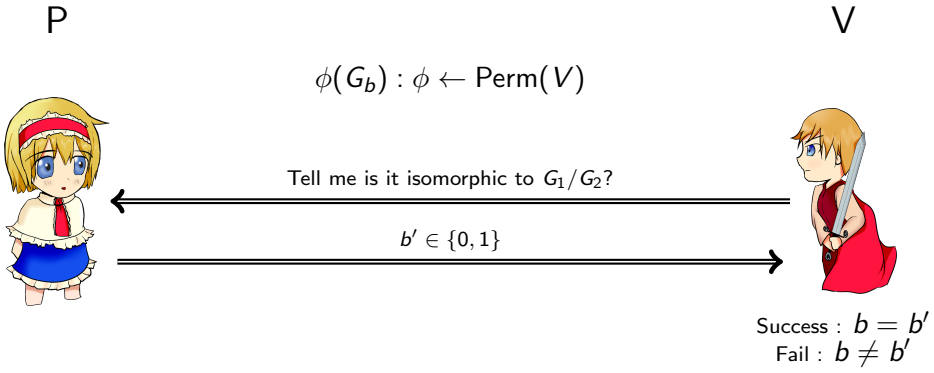

- Complete and Soundness error 2^{-t}
- Trival to see that NP $\subseteq$ IP
- IP = PSPACE

II. (Zero) Knowledge

The Notion of Knowledge

What is Knowledge ?

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Knowledge is the ability to complete a new task – Rafael Pass and Abhi Shelat

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- Hard to quantify knowledge.

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Can we quantify knowledge ?

- Hard to quantify knowledge.
- However, to quantify knowledge inherent in a message m, it is sufficient to quantify how much easier it becomes to compute a "new" function given m

The Notion of Knowledge– An Example

RSA Cryptosystem

- $N = pq, \phi(N) = (p-1)(q-1)$
- choose $e \in \mathbb{Z}_N^*$, $d = e^{-1} \mod \phi(N)$
- $pk_A = (N, e), sk_A = (N, d)$

A

B

$(m =$ "Meeting @ 10:30 p.m" $, \text{Sign}_{sk_A}(m))$

The Notion of Knowledge– An Example

I know the factorization of N

A

$(m =$ " Meeting @ 10:30 p.m" , $\text{Sign}_{sk_A}(m))$

B

The Notion of Knowledge– An Example

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I don't believe you

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I know the factorization of N

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(p, q)

Compute $sk_A = (N, d)$

A

B

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The Notion of Knowledge– An Example

A

B

The Notion of Knowledge– An Example

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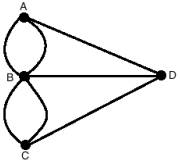
$(m =$ "You are fired" $, \mathrm{Sign}_{sk_B}(m))$

$(m =$ "You Bastard! Get out of this office" $, \mathrm{Sign}_{sk_A}(m))$

The Notion of Knowledge– An Example



"I know the graph is eulerian"

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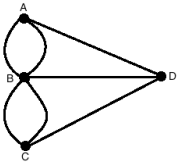

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Does the message convey any knowledge ?

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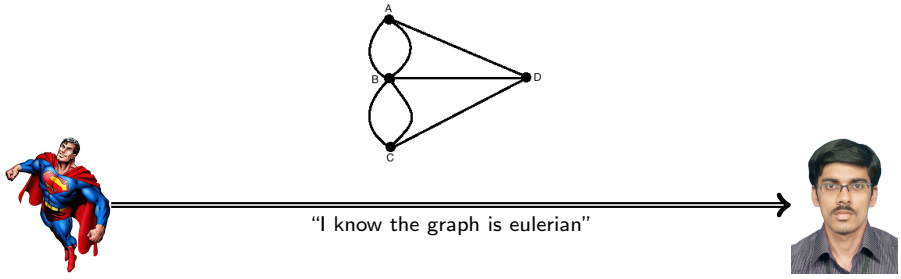

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No! I can very well compute whether the graph is "eulerian" or not.

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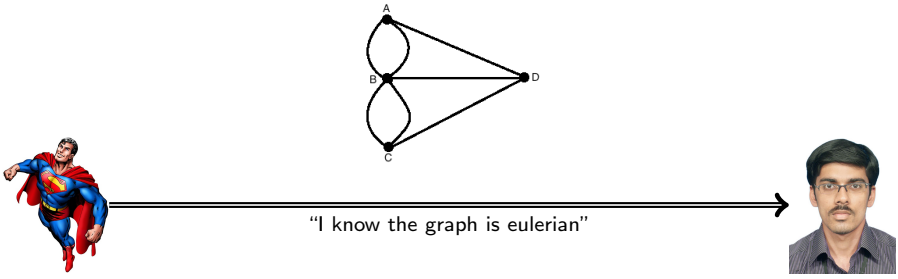

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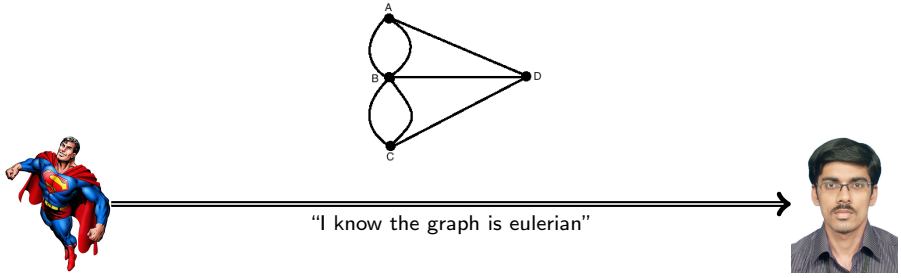

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Information which conveys no knowledge is called **zero knowledge**

III. (I) + (II) $\Rightarrow$ Zero Knowledge Proof.

Notion of Zero Knowledge Proof (ZKP)

Can a classical proof system be a zero knowledge proof system ?

Notion of Zero Knowledge Proof (ZKP)

Can a classical proof system be a zero knowledge proof system ?

It must be an interactive proof system

- Efficient- Verification of the proof should be simple – (V is a ppt algorithm)
- Completeness - $\forall x \in \mathcal{L}, \Pr[\text{V accepts in (P,V)}(x)] = 1$
- Soundness - $\forall x \notin \mathcal{L}, \forall \text{P}^*, \Pr[\text{V accepts in } (\text{P}^*, \text{V})(x)] \leq 1/2$

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How do we model the proof system does not convey any knowledge ?

Notion of Simulator

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- (P, V) is a ZKP system for $\mathcal{L}$, if $\forall$ PPT $V^*, \exists$ a PPT M^* such that $\forall x \in \mathcal{L}$

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$(P, V^*)(x)$ is the output of V^* after interacting with P on the common input $x \in \mathcal{L}$

Notion of Simulator

Rationale of Simulator

- It postulates that whatever a party can do "efficiently" by itself cannot be considered a gain from interaction with the outside.
- However, failure to provide a simulation of an interaction does NOT necessarily mean that this interaction results in some "real gain" (in some intuitive sense).
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Example: Eulerian Graph problem has a trivial ZKP

Variants of Zero Knowledge Proofs

Perfect Zero Knowledge Proof

(P, V) is PZK for $\mathcal{L}$ if $\forall \mathrm{V}^*$, $\exists M^*$ s.t. $\forall x \in \mathcal{L}$ the following two conditions hold:

- $Pr[M^\star(x) = \perp] \leq 1/2$,
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Statistical Zero Knowledge

(P, V) is SZK for $\mathcal{L}$ if $\forall \mathsf{V}^*$, $\exists M^*$ s.t. $\forall x \in \mathcal{L}$ the following two conditions hold: ensembles are statistically close as functions of $|x|$:

- $Pr[M^*(x) = \perp] \leq 1/2$,
- $\{(\mathsf{P}, \mathsf{V}^*)(x)\}_{x \in \mathcal{L}} \approx_{sd} \{M^*(x)\}_{x \in \mathcal{L}}$.

Variants of Zero Knowledge Proof

Computational Zero Knowledge

(P, V) is CZK for $\mathcal{L}$ if $\forall V^*$, $\exists M^*$ s.t. $\forall x \in \mathcal{L}$ the following two conditions hold: are computationally indistinguishable:

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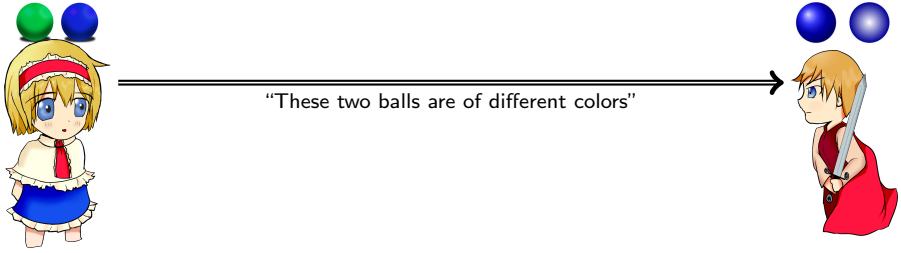
Relation among the notions

$$\boxed{BPP \subset PZK \subseteq SZK \subseteq CZK \subseteq IP}$$

Examples of ZKP : Two Balls and the Color-Blind Friend



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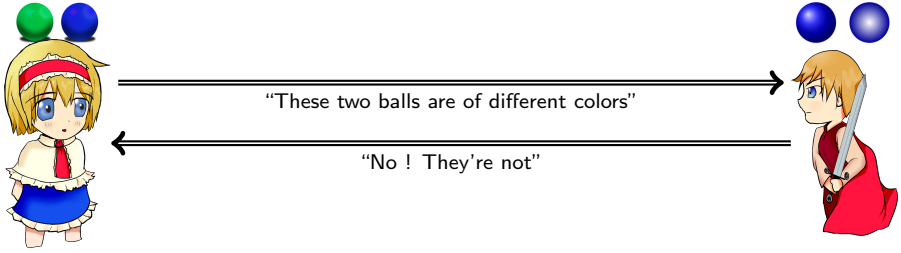

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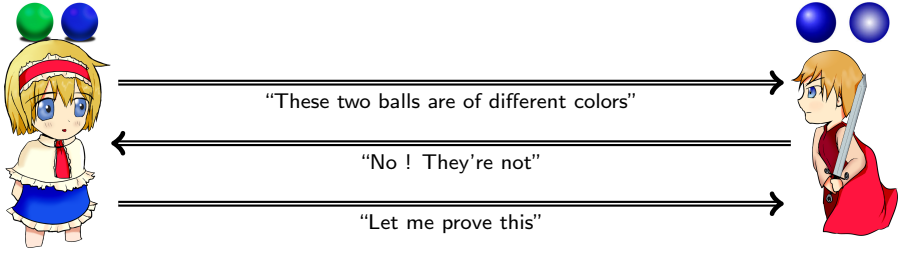

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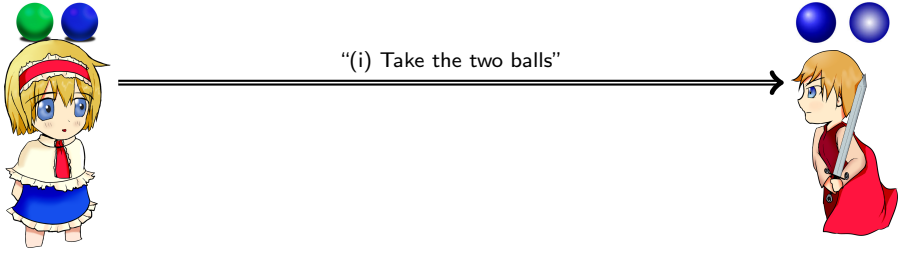

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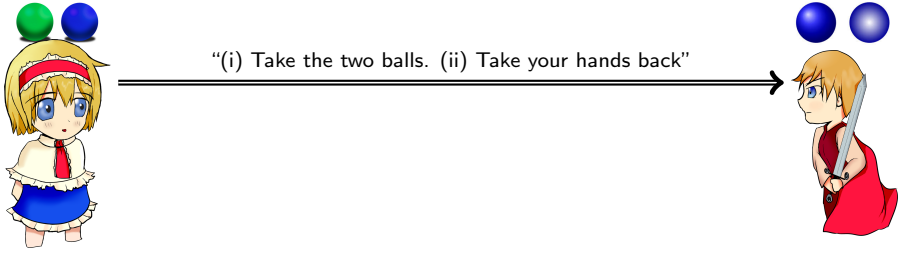

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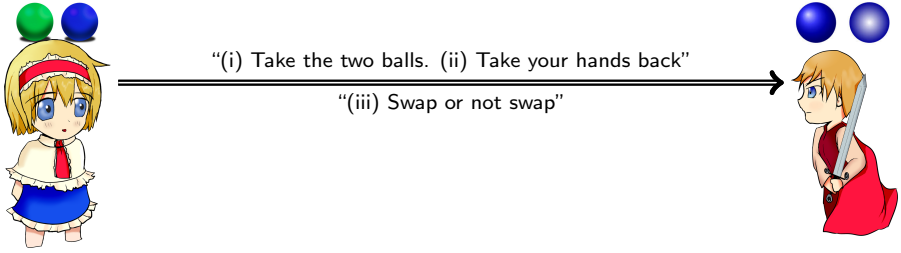

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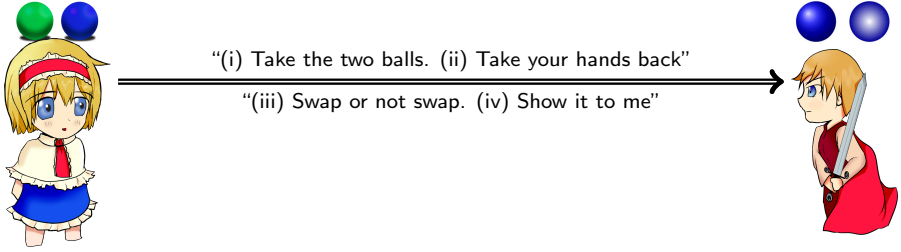

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swap if $b = 0$
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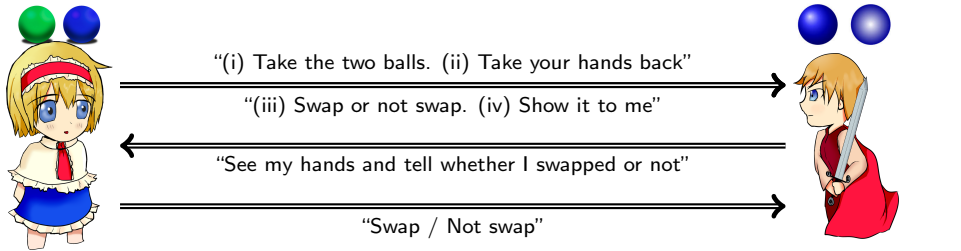

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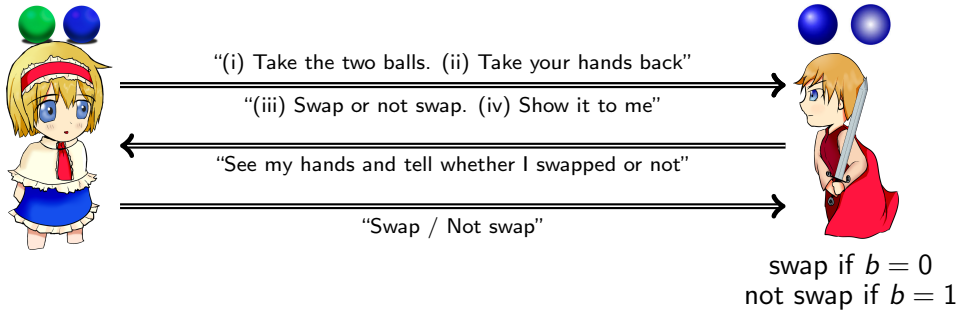

- If Alice really knows the balls are distinguishable, then she always wins – (**Complete**)
- If Alice does not know then she fails with probability 2^{-t} after 't' many repetitions – (**Soundness error**)

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After the experiment, Bob does not know which ball is of which color – (**Zero Knowledge**)

Example of ZKP : Discrete Log Problem

Recap of Discrete Log Problem

- G be a multiplicative cyclic group, generated by $\langle g \rangle \in G$.
- Each element $y \in G$ can be uniquely written as $y = g^x$ for some x
- (DL Problem) – Given g and $y \xleftarrow{\$} G$ find x

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Result

- Different sub-exponential algorithms are available, e.g., Pohlig-Hellman, index calculus etc.
- Nevertheless, the problem is hard to solve and the basis for Diffie-Hellman Key exchange algorithm

Example of ZKP : Discrete Log Problem

$$\langle g \rangle = \mathbb{Z}_p^*$$

$$x \in \mathbb{Z}_p^*, y \leftarrow g^x \mod p$$

(g, p, y)

Example of ZKP : Discrete Log Problem

" I want to access my locker"

Alice, (g, p, y)

Example of ZKP : Discrete Log Problem

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" I want to access my locker"

" Prove your indetity"

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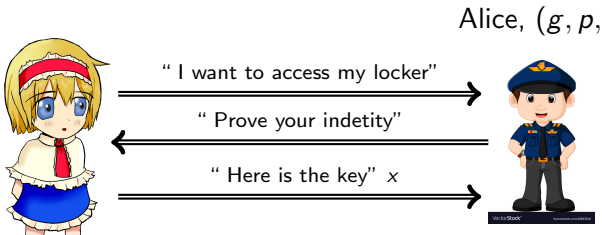

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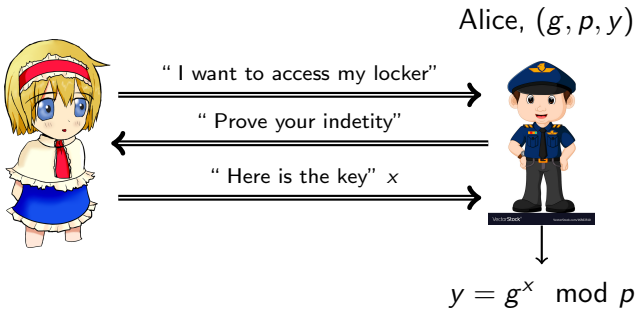


Example of ZKP : Discrete Log Problem

Alice, (g, p, y)

" Verified! You can access the locker"

Example of ZKP : Discrete Log Problem

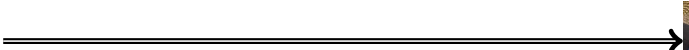
Vulnerability

"Here is the key of Alice, x"

Alice, (g, p, y)

Example of ZKP : Discrete Log Problem

Vulnerability



Example of ZKP : Discrete Log Problem

Vulnerability

Can Alice prove to the guard her secret x in zero knowledge way ?

Example of ZKP : Discrete Log Problem

$r \leftarrow \mathbb{Z}_p^*, h \leftarrow g^r \mod p$

h

Alice, (g, p, y)

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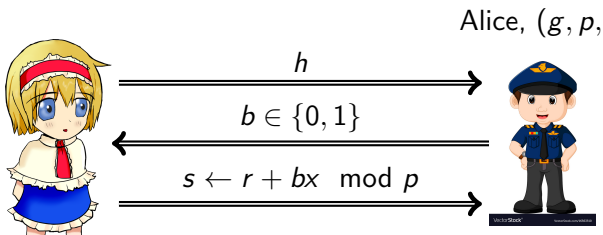

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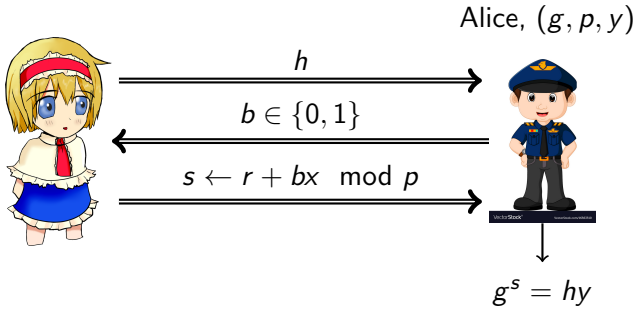

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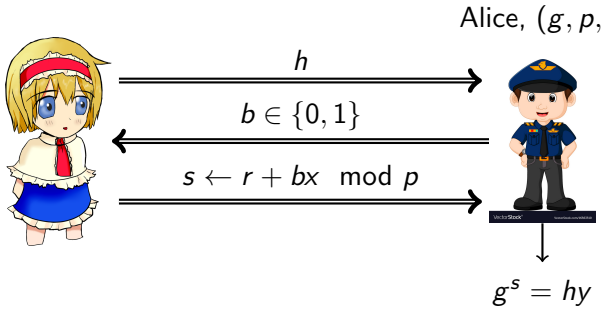

Repeat the game for 't' times

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Zero Knowledge: Defining the Simulator

view of the interaction $\text{view}^{\text{Alice}}_{\text{guard}^*}(g, p, y) = (h, b, s)$

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- Pick $b \xleftarrow{\$} \{0, 1\}$

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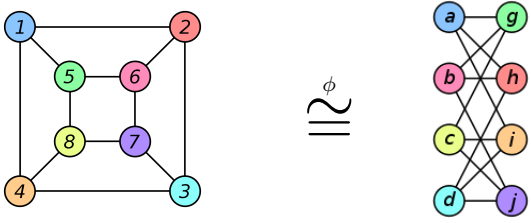
- Observe that $\text{view}^{\text{Alice}}_{\text{guard}^*}(g, p, y) \cong M^*(g, p, y)$ (**Try at home!**).

Example of ZKP : Graph Isomorphism (GI)



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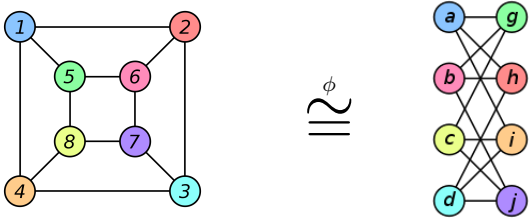

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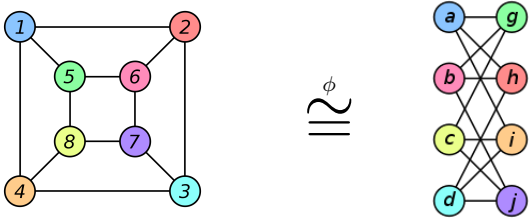

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However, to prove the statement, it requires an another tool called "'commitment scheme"

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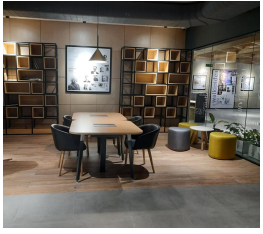
(... and continues)



Thank you for your kind attention !