Assignment 6

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1. A graph G is said to be a critical graph if $\chi(H) < \chi(G)$ for every proper subgraph H of G. G is said to be a k-critical graph if G is k-chromatic and critical.

   (a) Prove that for a k-critical graph G, $\delta(G) \geq k - 1$

   (b) Show that if a graph G has degree sequence $(d_1, d_2, \ldots, d_n)$ with $d_1 > d_2 > \ldots > d_n$, then $\chi < \max \min\{d_i + 1, i\}$.

2. Two player play a game on a graph G, alternatively choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceeding choices of the other player and hence they follow a path. The last person who is able to move, wins. Prove that the second player has a strategy to win if G has a perfect matching. Otherwise, first player wins.

3. You are the owner of a museum whose structure is depicted in Figure 3. What is the minimum number of CCTV cameras (also find the locations) that you will set up to ensure security of the interior of the museum? Assume that you can set up the CCTV camera only at the corners.

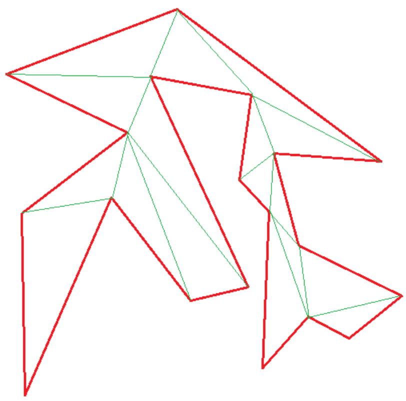

Figure 1: Structure of the Museum

4. An $m \times n$ integer matrix M is called a *magic rectangle* if it satisfies the following two properties:

   - $1 \leq M_{ij} \leq n$
   - No two entries in any row or in any column are equal.

The above two conditions immediately imply that $m \leq n$. For example, consider the following matrix which is a 3×5 magic rectangle

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \end{bmatrix}$$

Show that it is always possible to turn a magic rectangle to a magic square by inserting additional rows that satisfies the above two properties. (Hint: Convert the problem to a matching problem)

5. A graph G is outerplanar if G can be drawn in the plane so that all of its vertices lie on the exterior boundary.
   (a) Show that K_4 and $K_{2,3}$ are not outerplanar.
   (b) Deduce that, if G is an outerplanar graph, then G contains no subgraph homeomorphic or contractible to K_4 or $K_{2,3}$.
6. Given a tree T with n vertices, design an algorithm to color the edges of T using optimal number of colors.