Equivalent forms of LUB Property

Saikat Goswami

Department of Mathematics

Ramakrishna Mission Vidyamandira
Belur Math
shirsho112358@gmail.com

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Definitions

- **Ordered Field:**

  Let $\mathbb{F}$ be a nonempty set with two binary operations "$+$", "$\times$" and an order relation "$<$" defined on it such that:

  - $(\mathbb{F}, +)$ is an abelian group.
  - $(\mathbb{F} \setminus \{0\}, \times)$ is an abelian group.
  - $\times$ is distributive over $+$
  - If $a < b$ then $a + c < b + c$, for any $c \in \mathbb{F}$.
  - If $0 < a$ and $0 < b$ then $0 < ab$.

If $\mathbb{F}$ is nontrivial:

1. $-1_{\mathbb{F}} < 0_{\mathbb{F}} < 1_{\mathbb{F}}$
2. $m_{\mathbb{F}_{<n_{\mathbb{F}}}}$ iff $m < n$ for all $m, n \in \mathbb{Z}$.

- **Modulus Function on $\mathbb{F}$:**

$$|a-b|_{\mathbb{F}} := \begin{cases} a-b & ; a-b>0 \\ b-a & ; a-b<0 \end{cases}$$

Note:

1. $|a-b|_{\mathbb{F}} < \varepsilon \iff b-\varepsilon < a < b+\varepsilon$.
2. $|a+b|_{\mathbb{F}} < |a|_{\mathbb{F}} + |b|_{\mathbb{F}}$

- **Open Sets in $\mathbb{F}$:**

$S \subset \mathbb{F}$ is open if for any $x \in S$, $\exists\ \epsilon(>0) \in \mathbb{F}$ so that for every $y \in \mathbb{F}$ with $|y-x|_{\mathbb{F}} < \epsilon$ is also in S.

Note:

The open interval $(a,b)_{\mathbb{F}}$ is open.

- **Limit of a sequence in $\mathbb{F}$:**
  We say $\{x_n\} \underset{\text{seq.}}{\subseteq} \mathbb{F}$ converges to $x \in \mathbb{F}$ if for every $\epsilon(>0) \in \mathbb{F}$ there exists $k \in \mathbb{N}$ such that $|x_n - x|_{\mathbb{F}} < \epsilon$ for all $n \geq k$.

Note:

- The limit algebras (which holds in $\mathbb{R}$) also holds in $\mathbb{F}$.
- (x_n) converges to a unique limit.

- **Convergence of Series in $\mathbb{F}$:**
  $\sum_{n=1}^{\infty} a_n$ is said to be summable, if the sequence (s_n) converges in $\mathbb{F}$ where $(s_n) := \sum_{i=1}^{n} a_n$.

Note:

If $\sum_{n=1}^{\infty} a_n$ converges, then $(a_n) \to 0$.

- **Notion of Continuity in $\mathbb{F}$:**

  $D \subseteq \mathbb{F}$ with $f : D \to \mathbb{F}$. We say that f is continuous at $x \in D$ if for any $\varepsilon(> 0) \in \mathbb{F}$ there exists $\delta(> 0) \in \mathbb{F}$ such that for any $y \in D$ with $|y - x|_{\mathbb{F}} < \delta \implies |f(y) - f(x)|_{\mathbb{F}} < \varepsilon$

Sequential Criterion for Continuity:

- If f is continuous at $x \in D$ then for every sequence (x_n) in D that converges to x the sequence $f(x_n)$ converges to $f(c)$.
- In proving the converse one needs Archimedean Property, hence in an arbitrary Ordered Field $\mathbb{F}$ the converse may not be true.

- **Derivative of a function f in $\mathbb{F}$:**

  Let $f : (a, b) \to \mathbb{F}$ and let $c \in (a, b)$ then define $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ provided the limit exists.

Some Propositions from Analysis

A List of propositions in $\mathbb{F}$.

1. **Order Completeness Property:**
   If $S \subset \mathbb{F}$ is bounded above, then $\exists\ c \in \mathbb{F}$ that is an upperbound of S and for every upperbound b of S, we have $c \leq b$.

2. **Archimedean Property:**
   For every $x \in \mathbb{F}$, $\exists\ n \in \mathbb{N}_{\mathbb{F}}$ such that $x < n$.

3. **Cut Property:**
   - $A < B$, if every element of A is less than every element of B.
   - If $A, B \subset \mathbb{F}$ $(A \cap B = \phi\ ,\ A \cup B = \mathbb{F}\ ,\ A < B)$, then $\exists\ c \in \mathbb{F}\ (x \in A \implies x \leq c,\ x \in B \implies c \leq x)$
   - **Gap:** If $A, B \subset \mathbb{F}$ satisfies the hypothesis of Cut Property and violates its conclusion then we call A, B is a gap in $\mathbb{F}$.

4. **Topological Connectedness:**
   If $\mathbb{F} = A \cup B$ where A,B are nonempty and open, then $A \cap B \neq \phi$.

5. **Intermediate Value Property:**
   If $f : [a, b] \to \mathbb{F}$ be continuous with $f(a) < 0$ and $f(b) > 0$,then $\exists\ c \in (a, b)$ such that $f(c) = 0$.

6. **Bounded Value Property:**
   If $f : [a, b] \to \mathbb{F}$ is continuous, then $\exists B \in \mathbb{F}$ with $f(x) \leq B$ for all $x \in [a, b]$.

7. **Extreme Value Property:**
   If $f : [a, b] \to \mathbb{F}$ is continuous, then $\exists\ c \in [a, b]$ with $f(x) \leq f(c)\ \forall\ x \in [a, b]$

8. **Mean Value Property:**
   Suppose $f : [a, b] \to \mathbb{F}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

9. **Constant Value Property:**
Suppose $f : [a, b] \to \mathbb{F}$ is continuous on $[a, b]$ and differentiable on (a, b), with $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$

10. **Monotone Convergence Property:**
Every monotone bounded sequence in $\mathbb{F}$ is convergent in $\mathbb{F}$.

11. **Cauchy Completeness:**
Every Cauchy sequence in $\mathbb{F}$ is convergent in $\mathbb{F}$

12. **Fixed Point Property:**
$f : [a, b] \to [a, b]$ be continuous. Then there exists $x \in [a, b]$ such that $f(x) = x$

13. **Alternative Series Test:**
If $(a_n) \downarrow 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

14. **Absolute Convergence Property:**
If $\sum_{n=1}^{\infty} |a_n|$ converges in $\mathbb{F}$ then $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{F}$.

15. **Ratio Test Property:**
If $|\frac{a_{n+1}}{a_n}|_{\mathbb{F}} \to L \in \mathbb{F}$ with $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{F}$

16. **Nested Interval Property:**
If $I_1, I_2, ..., I_n, ...$ be a collection of closed and bounded nested intervals, i.e $I_1 \supseteq I_2 \supseteq ... \supseteq I_n \supseteq ...$ then $\bigcap_1^{\infty} I_i \neq \phi$

Now we move on to see which of these properties are equivalent to the Order Completeness in $\mathbb{F}$, and which of these are not.

FACT:

Any Ordered Field which is Order Complete is isomorphic to $\mathbb{R}$, so assuming Order Completeness in $\mathbb{F}$ will give us all the above stated propositions as Theorems in $\mathbb{F}$.

So to prove the equivalence of any above stated proposition with Order Completeness, all one only needs to see is, if or not that proposition gives back Order Completeness to us.

Cut Property $\iff$ Order Completeness

" $\Longleftarrow$ "

Immediate from Order Completeness.

" $\Longrightarrow$ "

We take any nonempty $S \subset \mathbb{F}$ bounded above, with B being the set of upperbounds of S, and A its complement.Now A,B satisfy the hypotheses of Cut Property,so there exists $c \in \mathbb{F}$ such that everything in A is less than c and everything in B is greater than c.

For here it's easy to check that c is a least upper bound of S.

Remark

This will be a very useful tool in the rest of the presentation.

Topological Connectedness $\iff$ Order Completeness

" $\implies$ " We prove the contrapositive statement.

Let A, B be a gap in $\mathbb{F}$.
$\Rightarrow \nexists\, c \in \mathbb{F}$ such that everything in A is less than c and everything in B is greater than c.
$\Rightarrow A$ has no maximum element and B has no minimum element.
Had such an element existed, then it would do the work of c, which contradicts the assumption.
$\Rightarrow$ both A and B are open, which contradicts the topological connectivity of $\mathbb{F}$.
Hence done.

Intermediate Value Property $\iff$ Order Completeness

" $\implies$ " Again we follow the method of contradiction.

Let A, B be a gap in $\mathbb{F}$. Now define a function $f : \mathbb{F} \to \mathbb{F}$ such that

$$f(x) = \begin{cases} 1 & x \in A \\ -1 & x \in B \end{cases}$$

You can shown that f is a continuous function, using the fact that both A and B has no max and min.

$\Rightarrow \nexists c \in \mathbb{F}$ with $f(c) = 0$ violating Intermediate Value Property.
Hence no such gap exists.

Extreme Value Property $\iff$ Order Completeness

" $\implies$ "
Let A, B be a gap in $\mathbb{F}$ with $a \in A$ and $b \in B$.

Define $f : [a, b] \to \mathbb{F}$ as follows:

$$f(x) = \begin{cases} x & x \in A \\ 0 & x \in B \end{cases}$$

f is continuous on $[a, b]$.But $\nexists\ c \in [a, b]$ with $f(x) \leq f(c)$ for all $x \in [a, b]$. (If such a c exists then it would have to be in A and A has no maximum element)Contradicting Extreme Value Property.

Hence no such gap exists in $\mathbb{F}$, which means $\mathbb{F}$ is order complete.

Mean Value Property $\iff$ Order Completeness

" $\implies$ "

- **Mean Value Property $\implies$ Constant Value Property**:

  Assumptions: $f : [a, b] \to \mathbb{F}$ which is continuous and differentiable on (a, b) with $f'(x) = 0$ for all $x \in (a, b)$

  Claim: f is constant on $[a, b]$

  Pf: Apply MVP to $f : [a, x] \to \mathbb{F}$ where $x \in (a, b]$ we get $c \in [a, x]$ with

$$0 = f'(c) = \frac{f(x) - f(a)}{x - a} \implies f(x) = f(a) \text{ for all } x \in (a, b].$$

  i.e f is constant in $[a, b]$

- **Constant Value Property $\implies$ Order Completeness**

Let A, B be a gap in $\mathbb{F}$. Consider the function $f : \mathbb{F} \to \mathbb{F}$ defined as

$$f(x) = \begin{cases} 1 & x \in A \\ -1 & x \in B \end{cases}$$

Want: $f'(c) = 0$ for all $c \in \mathbb{F}$. i.e
$\forall\ \varepsilon(> 0) \in \mathbb{F}$, $\exists\ \delta > 0$ s.t $\forall\ x \in \mathbb{F}$ $\left(|x - c|_{\mathbb{F}} < \delta \Rightarrow \left| \frac{f(x)-f(c)}{x-c} \right|_{\mathbb{F}} < \varepsilon \right)$.

WLOG, lets assume $c \in A$. Since A, B is a gap, A has no maximum element. So let $a \in A$ with $c < a$.
Choose $\delta = a - c$. If $x \in \mathbb{F}$ with $|x - c|_{\mathbb{F}} < \delta$ then $c - \delta < x < c + \delta$

$$\implies x < a \implies x \in A \implies \left| \frac{f(x)-f(c)}{x-c} \right|_{\mathbb{F}} = 0 < \varepsilon.$$

Hence $f'(c) = 0$ for any $c \in \mathbb{F}$.So f has derivative 0 everywhere, yet it is'nt constant on $[a, b]$ if one takes $a \in A$ and $b \in B$. Contradiction.

Monotone Convg. Property $\iff$ Order Completeness

- Monotone Convergence Property $\implies$ Archimedean Property.

  Let's assume $\mathbb{F}$ is not archimedean, $\exists\ c \in \mathbb{F}$ with $n < c$ for all $n \in \mathbb{N}_{\mathbb{F}}$.
  By assumption $(1, 2, ...)$ must converge,say to $r \in \mathbb{F}$.
  $\implies$ $(0, 1, 2, ...)$ also converges to r.
  Subtracting the two sequence we find (1,1,1,...) converges to 0, which is absurd.
  Therefore $\mathbb{F}$ must be Archimedean.

- Let $(\phi \neq)A \subseteq \mathbb{F}$ which is bounded above in $\mathbb{F}$. U: Set of upperbounds of A in $\mathbb{F}$.

  Want: U has a minimum element.

- **Claim 1:** $\{u - \varepsilon : u \in U\} =: U - \varepsilon \not\subseteq U$, for all $\varepsilon(> 0) \in \mathbb{F}$.

  Let $\varepsilon > 0$.
  Let $U - \varepsilon \subseteq U$,Now we use induction on $n \in \mathbb{N}_{\mathbb{F}}$.
  If $U - n\varepsilon \subseteq U$ for some $n \in \mathbb{N}_{\mathbb{F}}$.
  then $U - (n+1)\varepsilon = (U - \varepsilon) - n\varepsilon \subseteq U - n\varepsilon \subseteq U$.
  $\implies U - n\varepsilon \subseteq U$ for all $n \in \mathbb{N}_{\mathbb{F}}$.
  Hence by Archimedean Property we have $\mathbb{F} = \bigcup_{n=1}^{\infty} U - n\varepsilon \subseteq U$, which contradicts $A \neq \phi$.

- **Claim 2:** $\bigcap_{n=1}^{\infty} U - \frac{1}{n} \subseteq U$.

  Let $x \in \bigcap_{n=1}^{\infty} U - \frac{1}{n}$and let $(x <)y \in \mathbb{F}$.
  By archimedean property $\exists\ n \in \mathbb{N}_{\mathbb{F}}$ such that $x + \frac{1}{n} < y$.
  $x \in U - \frac{1}{n} \Rightarrow x + \frac{1}{n} \in U \Rightarrow (x + \frac{1}{n} <)y \notin A$.
  y is chosen arbitrarily,so x is an upperbound of A, i.e $x \in U$.

- **Claim 3:** $U - \frac{1}{n} \subseteq U - \frac{1}{m}$ for all $m \leq n$.

- **Claim 3.1:** Let n_k be any increasing sequence in $\mathbb{N}_{\mathbb{F}}$ then $\bigcap_{k=1}^{\infty} \left(U - \frac{1}{n_k}\right) = \bigcap_{n=1}^{\infty} \left(U - \frac{1}{n}\right) \subseteq U$.

- **Claim 4:** If $x \in \mathbb{F}$ with $x \notin U - \frac{1}{n}$, then $x < y$ for all $y \in U - \frac{1}{n}$.

- For a detailed proof see [1]

- We proceed in constructing a monotone increasing sequence, to apply M.C.P. The limit will be the desired lub A.

- Let $x_1 \in (U-1)\setminus U$ ($\neq \phi$ by Claim 1)
  $x_1 \notin U \Rightarrow x_1 \notin \bigcap_{n=1}^{\infty} U - \frac{1}{n}$.(Claim 2)
  so $\exists\ n_1 > 1$ such that $x_1 \notin U - \frac{1}{n_1}$.
  Let $x_2 \in \left(U - \frac{1}{n_1}\right) \setminus U$, $\exists\ n_2 > n_1$(Claim 3)
  such that $x_2 \notin U - \frac{1}{n_2}$.
  Again consider $x_3 \in \left(U - \frac{1}{n_2} \setminus U\right)$; and so on.
- This yields two increasing sequences:
  $(1, n_1, n_2, ...)$ in $\mathbb{N}_{\mathbb{F}}$ (Claim 3) and (x_k) in $\mathbb{F}$ (Claim 4,3)
  such that $x_k \in \left(U - \frac{1}{n_{k-1}}\right) \setminus U$.
  Then (x_k) is bounded above by each element of U.
  $\implies (x_k) \to x \in \mathbb{F}$.
  $\implies x \leq u$ for all $u \in U$.
- $x_k \leq x$ for all k and $x_k \in \left(U - \frac{1}{n_k}\right)$
  $\implies x \in \bigcap_{k=1}^{\infty}\left(U - \frac{1}{n_k}\right) = \bigcap_{n=1}^{\infty}\left(U - \frac{1}{n}\right) \subseteq U$, i.e $x \in U$.
  Therefore x is the smallest element in U, which means $x = \text{lub}\, A$.
  Hence $\mathbb{F}$ is order complete.

Fixed Point Property $\iff$ Order Completeness

"$\implies$"

Let A, B be a gap in $\mathbb{F}$. Pick $a \in A$ and $b \in B$ and define $f : [a, b] \to [a, b]$ such that

$$f(x) := \begin{cases} b & x \in A \\ a & x \in B \end{cases}$$

A has no maximum element and B has no minimum element,we can show that f is continuous, with no fixed point. Contradicts Fixed Point Property.

Hence $\mathbb{F}$ must be Order Complete.

Nested Interval Property $\underset{\text{Archimedian}}{\Longleftrightarrow}$ Order Completeness

" $\underset{\text{Archimedian Prop.}}{\Longrightarrow}$ "

$(\phi \neq)A \underset{bdd.}{\subseteq} \mathbb{F}$. **Want:** $\text{lub}\, A \in \mathbb{F}$ using NIP.
Create a nested sequence of closed bounded intervals.

b_0: an upperbound of A, a_0: not an upperbound of A.
Then define $I_0 := [a_0, b_0] \Rightarrow I_0 \cap A \neq \phi$.
Choose $c = (a_0 + b_0)/2_{\mathbb{F}}$.
If $\exists\ a \in A$ with $c < a$ choose $a_1 = c$, $b_1 = b_0$, otherwise $a_1 = a_0$, $b_1 = c$.
We created $I_0 \supseteq I_1 \supseteq ... \supseteq I_n \supseteq ...$ where $I_n = [a_n, b_n]$ with
a_n: not an upperbound of A, b_n: upperbound of A.

$\text{diam}(I_n) = \frac{b_0 - a_0}{2^n}$

Assuming $\mathbb{F}$ is Archimedean.

$(\frac{1}{2^n}) \to 0 \Rightarrow \text{diam}(I_n) \to 0 \Rightarrow \bigcap_1^\infty I_i$ contain at most 1 element.
NIP $\Rightarrow$ It contains 1 element, say $\bigcap_1^\infty I_i = \{s\}$.

Want: $s = \text{lub}\, A$.
If $\exists\ a \in A$ with $s < a \Rightarrow a - s > 0$.
$\text{diam}(I_n) \to 0\ \exists\ p \in \mathbb{N}$, $\text{diam}(I_p) < a - s$.
$s \in I_p$ and $a \notin I_p \Rightarrow b_p < a$. Contradicts b_p: upperbound of A.
$\Rightarrow a \leq s$. i.e s is an upperbound of A.

Let $k < s$.$\exists\ q \in \mathbb{N}$, $\text{diam}(I_q) < s - k$.
$s \in I_q$ and $k \notin I_q \Rightarrow k < a_q$. Way we have constructed a_q
$\exists\ b \in A$ with $k < a_q < b \Rightarrow k$ is not an upperbound of A.
$s = \text{lub}\, A$.

Remark:

In general NIP $\not\Rightarrow$ Order Completeness. Hyperreal Numbers forms a counterexample. See [5]

Ratio Test Property $\iff$ Order Completeness

- **Ratio Test $\Rightarrow$ Archimedean Property:**

  Consider the following sequence $\left(\frac{1}{2^n}\right)$then $\left(\left|\frac{a_{n+1}}{a_n}\right|\right) \to \frac{1}{2} < 1$.
  $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.i.e sequence of partial sums
  $(s_n) = \left(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right)$ is convergent.

  If $\mathbb{F}$ is non archimedean, $\exists \varepsilon (> 0) \in \mathbb{F}$ with $\varepsilon < \frac{1}{n}$ for all $n \in \mathbb{N}_{\mathbb{F}}$.

  For any $n \in \mathbb{N}$, $\varepsilon < |s_{n+1} - s_n| = \frac{1}{2}$ i.e (s_n) is not Cauchy, contradiction.
  So $\mathbb{F}$ must be Archimedian.

- **Every Archimedean Ordered Field $\mathbb{F}$ can be embedded in $\mathbb{R}$:**

1 **Defining Rational Copies in $\mathbb{F}$:** $\phi : \mathbb{Q} \to \mathbb{F}$ mapping $\frac{m}{n} \to \frac{m_{\mathbb{F}}}{n_{\mathbb{F}}}$.

ϕ is a well defined, $(1-1)$ Order-preservingField homomorphism.

2 $x \in \mathbb{F}$.Define $S_x := \{q \in \mathbb{Q} : q_{\mathbb{F}} < x\} \subset \mathbb{R}$.

$\mathbb{F}$ is Archemedean $\Rightarrow S_x$ is bounded and nonempty.

3 **Define** $\Phi : \mathbb{F} \to \mathbb{R}$ with $\Phi(x) = \text{lub}\, S_x$.

Φ is a well defined, Order Preserving Field homomorphism.

- **Subfield of $\mathbb{R}$ that satisfies the Ratio Test must contain every reals:**

  $\mathbb{F}$: Archimedean Subfield of $\mathbb{R}$ where Ratio Test holds.
  It's enough to prove $\mathbb{F} = \mathbb{R}$.

  $n \pm \frac{1}{2} \pm \frac{1}{4} \pm \frac{1}{8} \pm ...$ is a series whose terms are in $\mathbb{F}$and Ratio Test confirms the convergence of this series in $\mathbb{F}$.

  It suffices to show for $x \in \mathbb{R}$ there is a series of the form $\left(n \pm \frac{1}{2} \pm \frac{1}{4} \pm \frac{1}{8} \pm ...\right) \to x \Rightarrow x \in \mathbb{F}$.

- Its enough to consider $x \in [0,1]$. because the representations of other reals can be obtained by adding an integer.

2
Let $x \in [0,1]$. Bisect $[0,1]$,choose $a_1 := \frac{1}{2}$.

Let $x \neq \frac{1}{2}$, if $x \in \left(0, \frac{1}{2}\right)$ choose $a_2 := \frac{1}{2} - \frac{1}{4}$ otherwise $a_2 := \frac{1}{2} + \frac{1}{4}$.
Continuing this way with the help of N.I.P we can conclude that $(a_n) \to x$, where $(a_n) = (\frac{1}{2} \pm \frac{1}{4} \pm \frac{1}{8} \pm ...)$

So $x \in \mathbb{F}$

$\Rightarrow \mathbb{F} = \mathbb{R} \Rightarrow \mathbb{F}$ is Order Complete.

Example of a Non-Archimedean Ordered Field:

Let $\mathbb{F}$ be any ordered field.

Define $\mathbb{F}((x)) := \{ \sum_{k=-n}^{\infty} a_k x^k : a_k \in \mathbb{F} \;\; n \in \mathbb{N}\}$.

For any element of $\mathbb{F}((x))$, the associated finite sum $\sum_{k=-n}^{-1} a_k x^k$ is called its principal part.

- **Addition:** $\sum_{k=-n} a_k x^k + \sum_{k=-m} b_k x^k := \sum_{k=-max\{n,m\}} (a_n + b_n) x^n$
- **Multiplication:** $\sum_{k=-n} a_k x^k \times \sum_{k=-m} b_k x^k := \sum_{k=-(n+m)} \left(\sum_{i+j=k} (a_i b_j) \right) x^k$
- **Order Relation:** $\left(\sum_{k=-n}^{\infty} a_k x^k \right) \geq 0$ iff the first nonzero coefficient is greater or equal to 0. Using this we can define $\alpha \leq \beta$ iff $0 \leq \beta - \alpha$.

  This makes $(\mathbb{F}((x)), +, \times, \leq)$ an ordered field.

- **Archimedean Property:**

  $x^{-1} \in \mathbb{F}((x))$ with $x^{-1} - n > 0$ for any $n \in \mathbb{N}_{\mathbb{F}}$.
  $\Rightarrow n < x^{-1}$ for all $n \in \mathbb{N}_{\mathbb{F}}$.

  $x \in \mathbb{F}((x))$, for any $n \in \mathbb{N}_{\mathbb{F}}$ we have $0 < 1/n - x$.
  $\Rightarrow x < 1/n$ for all $n \in \mathbb{N}_{\mathbb{F}}$

  $\Rightarrow \mathbb{F}$ is **non-archimedean** $\Rightarrow \mathbb{F}$ **is not Order Complete**.

- **Modulus Function:**

$$\left| \sum_{k=-n} a_k x^k \right| := \begin{cases} \sum_{k=-n} a_k x^k & \text{if first nonzero coefficient is postive} \\ \sum_{k=-n} (-a_k) x^k & \text{if first nonzero coefficient is negetive} \end{cases}$$

- **Cauchy Sequences:**(a_n) is Cauchy,
  If $\forall\ (0<)\varepsilon \in \mathbb{F}((x))$ there exists $k \in \mathbb{N}$
  $\big(\forall n, m \geq k\ (|a_n - a_m|_{\mathbb{F}((x))} < \varepsilon)\big)$

Note:

The sequence $(x^n) \downarrow 0$, whereas the sequence $(\frac{1}{n}) \nrightarrow 0$. So here in $\mathbb{F}((x))$, (x^n) will play the role which $(\frac{1}{n})$ plays in $\mathbb{R}$.

- **Characterizing Cauchy Sequences in $\mathbb{F}((x))$:**

  (a_n) be Cauchy in $\mathbb{F}((x))$.
  $\varepsilon = x^k$ then $\exists\ N_k \in \mathbb{N}$ with:
  $|a_n - a_m|_{\mathbb{F}((x))} < x^k$ for any $n, m \geq N_k$.
  The first nonzero term of $|a_n - a_m|_{\mathbb{F}((x))}$ is positive.
  If the first nonzero term corresponds to x^j with $j < k$.
  $\Rightarrow x^k < |a_n - a_m|$, contradiction.
  $\Rightarrow$ Coefficients of x^j with $j < k$ in $a_n - a_m$ are 0 for all $n, m \geq N_k$.

- A sequence of Formal Laurent Series is Cauchyiff the sequence of its principal part stabilizes (is eventually constant) and for every integer $n > 0$ the sequence of coefficients of x^n stabilizes.

- See this sequence

$$\begin{aligned} a_1 &= \qquad\qquad\qquad x^{-1} + 1 + x + x^2 + \ldots \\ a_2 &= \qquad\quad x^{-2} + x^{-1} + 1 + x + x^2 + \ldots \\ a_3 &= x^{-3} + x^{-2} + x^{-1} + 1 + x + x^2 + \ldots \\ &\vdots \end{aligned}$$

is not Cauchy, even though for every integer n the sequence of coefficients of x^n is constant.

Note:

If $(a_n) = \left(\sum_{j \geq J_n} A_{n,j} x^j \right)$ is Cauchy, then $\min\{J_n : n \in \mathbb{N}\} = \mathfrak{J} \in \mathbb{Z}$.

- **Cauchy Completeness:** Let (a_n) be Cauchy.

  Writing $a_n = \sum_{j \geq J_n} A_{n,j} x^j$.
  If we fix j then for each j we get that $(A_{n,j})_{n \geq 1}$ is constant for n large enough and

  $$\lim_{n\to\infty} a_n = \sum_{j \geq \mathfrak{J}} (\lim_{n\to\infty} A_{n,j}) x^j \in \mathbb{F}((x))$$

  Hence $\mathbb{F}((x))$ is Cauchy Complete.

- **If $(a_n) \to 0$ then $\sum_{n=1}^{\infty} a_n$ is convergent:**

  $a_n = \sum_{j \geq J_n} A_{n,j} x^j$ with $(a_n) \to 0 \Rightarrow (a_n)$ is Cauchy. Then for each j

  $(A_{n,j})$ is eventually 0.$\Rightarrow \sum_{n=1}^{\infty} A_{n,j} (= A_j)$ terminates after finitely many steps.

  $$\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{j \geq \mathfrak{J}} \left(\sum_{n=1}^{\infty} A_{n,j} \right) x^j = \sum_{j \geq \mathfrak{J}} A_j x^j \in \mathbb{F}((x)).$$

- **If** $\sum_{n=1}^{\infty} |a_n|_{\mathbb{F}((x))}$ **converges then** $\sum_{n=1}^{\infty} a_n$ **also converges:**

$\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow |a_n| \to 0$

$\Rightarrow$ For any $\varepsilon(> 0) \in \mathbb{F}((x))$

$\exists\ N \in \mathbb{N}$ with $||a_n| - 0| < \varepsilon$ for any $n \geq N$

$\Rightarrow |a_n - 0| < \varepsilon$ for any $n \geq N$

$\Rightarrow (a_n) \to 0.$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

Cauchy Completeness $\not\Rightarrow$ Order Completeness:

$\mathbb{F}((x))$ is Cauchy Complete but not Order Complete.

It has always been very tempting to think that these two were equivalent in the presence of the order field axioms.

It's worth knowing what Hilbert had in mind when he referred $\mathbb{R}$ as the "Complete Archimedean Ordered Field". He meant that:

Every Archimedian Ordered Field is isomorphic to a subfield of "the" Order Complete Field ($\mathbb{R}$)

What Hilbert meant by $\mathbb{R}$ **is complete** is: Nothing can be added to $\mathbb{R}$ to make a larger ordered field, without sacrificing the Archimedian Property.

Other Properties $\nRightarrow$ Order Completeness:

- **Alternating Series Test $\nRightarrow$ Order Completeness**

  In $\mathbb{F}((x))$ every series whose terms tend to 0(whether or not they alternate in sign) is summable, so Alternating Series Test holds in $F((x))$ even though it is not Order Complete.

- **Absolute Convergence Property $\nRightarrow$ Order Completeness**

  In $\mathbb{F}((x))$ Absolute Convergence Property holds but is not Order Complete.

- **Bounded Value Property $\nRightarrow$ Order Completeness**

  A counterexample can be found in [1.] listed in the Reference.

Conclusion:

Propositions $\Leftrightarrow$ Order Completeness:

- Order Completeness
- Cut Property
- Topological Connectedness
- Intermediate Value Property
- Extreme Value Property
- Mean Value Property
- Constant Value Property
- Monotone Convergence Property
- Fixed Point Property
- Ratio Test Property

Propositions $\not\Leftrightarrow$ Order Completeness:

- Archimedean Property
- Bounded Value Property
- Cauchy Completion
- Alternative Series Test
- Absolute Convergence Property
- Shrinking Interval Property
- Nested Interval Property

References:

1. MCT iff Order Completion, available at
   https://math.stackexchange.com/questions/3988327

2. James Propp, Real Analysis in Reverse, available at
   http://faculty.uml.edu/jpropp/reverse.pdf

2. J.F Hall, Completeness of Ordered Fields, available at
   https://arxiv.org/pdf/1101.5652.pdf

3. R.G Bartley and D.R Sherbert, Introduction to Real Analysis.

4. H Jerome Keisler, Foundations of Infinitesimal Calculus, available at
   https://1lib.in/book/2030517/baf738

5. William S. Hatcher, Calculus is Algebra, Amer. Math. Monthly 89(1982) pp.362-370, available at https://www.jstor.org/stable/2321645?seq=1#metadata_info_tab_contents

6. Ordered Fields are not finite. https://math.stackexchange.com/questions/156554

7. Do there exists any Order Complete Field at all? https://math.stackexchange.com/questions/3988465