# Prime Numbers & Irreducible Polynomials

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# Overview

1. **Buniakowski Conjecture**
   - Investigating the assumptions in the Conjecture
   - Extracting an Irreducibility criterion from the Conjecture
2. **Murty's Irreducibility Criterion (M.I.C.)**
   - Root Capturing Lemma (1)
   - Proving Murty's Irreducibility Criterion
   - Generalizing Murty's Irreducibility Criterion
3. **Constructing Irreducible Poly. using Prime numbers**
   - Cohn's Irreducibility Criterion
   - Root Capturing Lemma (2)
   - Proving Cohn's Irreducibility Criterion

# Section 1

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## Example:

$f(x) = x^2 + 1 \implies (f(n))_{n\in\mathbb{N}}$ contains primes infinitely often (i.o.).

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**Dirichlet's Prime Number Theorem:**

$a, d \in \mathbb{N}$ with $(a, d) = 1$ there are infinitely many primes of the form $a + nd$ where $n$ is a positive integer.

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If $f$ were reducible with $f(x) = g(x)h(x)$. Then $f(n) = g(n)h(n)\ \forall n \in \mathbb{N}$. So $g(x)$ and $h(x)$ takes the value $\pm 1$ infinitely many times ; Contradiction! So $f(n)$ is composite for all large $n$.

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Do (1) & (2) $\implies$ there are primes infinitely often in $(f(n))_{n\in\mathbb{N}}$ ?

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So the $3rd$ assumption is very crucial in the generation of primes infinitely often in $(f(n))_{n\in\mathbb{N}}$.

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If $\{f(n) : n \in \mathbb{N}\}$ has a common divisor $d > 1 \implies d \mid f(n)\ \forall\ n \in \mathbb{N}$.
Hence $(f(n))_{n\in\mathbb{N}}$ cannot contain primes i.o.

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**A stronger version of the above criterion (M.I.C.)**

$(f(n))_{n\in\mathbb{N}}$ contains a prime for sufficiently large $n$ $\implies$ $f$ : irreducible

# Section 2

# Formal statement of M.I.C.

## Murty's Irreducibility Criterion (M.I.C.)

$f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[X]$ with

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$$\implies |\alpha|^m \leq H\left(|\alpha|^{m-1} + \cdots + |\alpha| + 1\right) = H\left(\frac{|\alpha|^m - 1}{|\alpha| - 1}\right)$$

$$\text{Then } |\alpha| > 1 \implies |\alpha|^{\mathbf{m}}\left(|\alpha| - \mathbf{1}\right) < H\,|\alpha|^m - H < \mathbf{H}\,|\alpha|^{\mathbf{m}}$$

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- Now we arrive at a contradiction (since $|g(n)| = 1$).

$$\mathbf{|g(n)|} \geq \prod_i (n - |\alpha_i|) \;>\; \prod_i (n - (H+1)) \geq \mathbf{1}$$

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# Section 3

# Irreducible Polynomials from Prime numbers

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## Cohn's Irreducibility Criterion:

$b \geq 2$ be any natural number and $p$ a prime with

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$$101 = 9^2 + 2 \cdot 9 + 2$$

$$101 = 10^2 + 1$$

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| | |
|---|---|
| $101 = 2^6 + 2^5 + 2^3 + 1$ | $f(x) = x^6 + x^5 + x^3 + 1$ |
| $101 = 3^4 + 2 \cdot 3^2 + 2$ | $f(x) = x^4 + 2x^2 + 2$ |
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| $101 = 2 \cdot 6^2 + 4 \cdot 6 + 5$ | $f(x) = 2x^2 + 4x + 5$ |
| $101 = 2 \cdot 7^2 + 3$ | $f(x) = 2x^2 + 3$ |
| $101 = 8^2 + 4 \cdot 8 + 5$ | $f(x) = x^2 + 4x + 5$ |
| $101 = 9^2 + 2 \cdot 9 + 2$ | $f(x) = x^2 + 2x + 2$ |
| $101 = 10^2 + 1$ | $f(x) = x^2 + 1$ |

## Cohn's Irreducibility Criterion

$$p = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 \quad \text{where } 0 \leq a_i \leq b-1$$

$$\implies f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \text{ is irreducible in } \mathbb{Z}[X].$$

## An important Note:

The positivity of the coefficient's plays a crucial role in Cohn's Theorem.

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The positivity of the coefficient's plays a crucial role in Cohn's Theorem.

$$\mathbf{101} = 10^3 - 9 \cdot 10^2 + 10 - 9 \quad \text{is a prime number}$$

but the corresponding $\mathbf{f(x) = x^3 - 9x^2 + x - 9} = (x-9)(x^2+1)$ is reducible in $\mathbb{Z}[X]$.

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$$a_n \geq 1 \ , \ a_{n-1} \geq 0 \text{ and } |a_i| < H \ \text{ for } \ i = 0, 1, ..., n-2$$

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where $H$ is some positive constant.

Then any complex zero of $f$ either has nonpositive real part or satisfies

$$|\alpha| < \frac{1 + \sqrt{1 + 4H}}{2} = K \ (say)$$

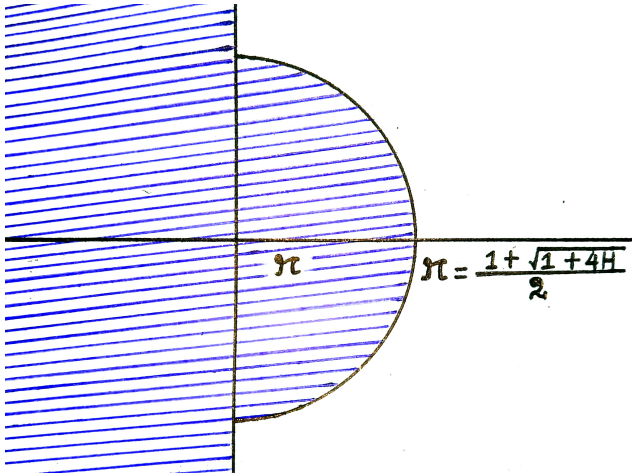


**Figure:** All the roots of $f(x)$ will lie in the shaded region.

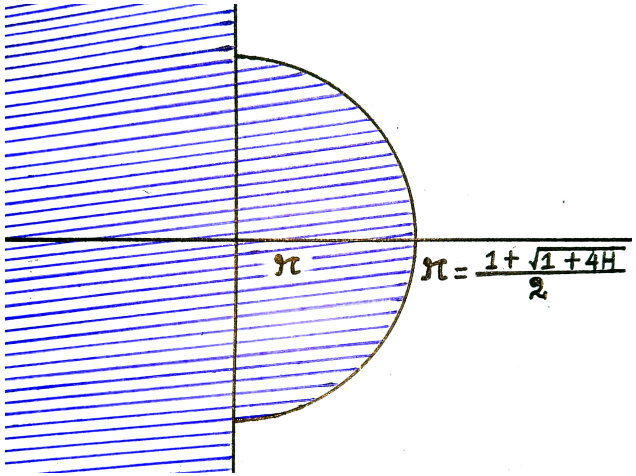


**Figure:** All the roots of $f(x)$ will lie in the shaded region.

## Strategy:

We show that for any complex number $z$ which lies

on the Right-half Plane & outside the Open Disc of radius $\frac{1+\sqrt{1+4H}}{2}$

we have $f(z) \neq 0$.

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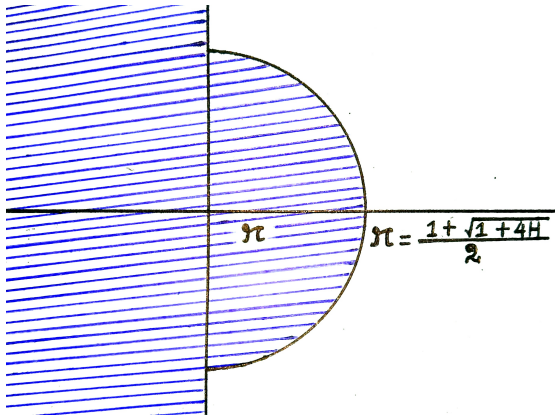
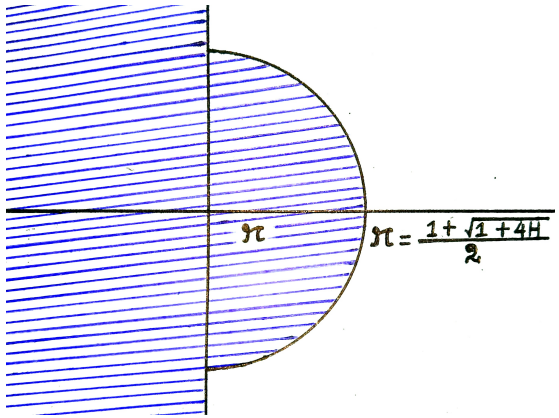
- Now $g$ is of the form

$$g(x) = c\prod_i (x - \alpha_i)$$

  $c$ : leading coefficient of $g$ & $\alpha_i$ runs over a subset of the zeros of $f$.




## Deploying the $2^{nd}$ Root Capturing Lemma

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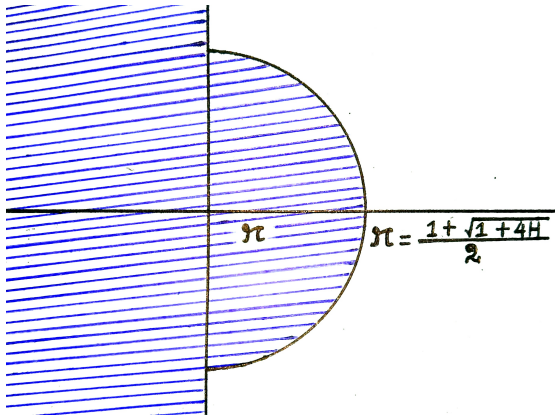
1) $\Re(\alpha) \leq 0$ ; i.e. $\alpha$ lies on the left half plane.

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OR

**2)** $|\alpha| < \frac{1+\sqrt{1+4(b-1)}}{2}$ ; i.e. If it lies on the right half plane, it lies inside the mentioned disc.

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The proof BREAKS DOWN in the highlighted inequality when $b = 2$

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- Then after investigating the properties of all the complex roots of $g(x)$ we will obtain the inequality $\mid g(2) \mid > 1$, giving a contradiction.
- Hence coming to the conclusion that $f(x)$ is irreducible for the case $b = 2$

# References

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Prime numbers and Irreducible Polynomials

*The American Mathematical Monthly;* 109:05, 452 — 458.

Kurt Gritzman (2005)

On an irreducibility criterion of M. Ram Murty

*The American Mathematical Monthly;* 112:3, 269 — 270.

M. A. Lee (1969)

Some irreducible polynomials which are reducible mod $p$ for all $p$

*The American Mathematical Monthly;* 76, 1125.

$f(x) = x^4 + (4a + 2)x + 1$ is reducible mod $p$ for all prime $p$.

`https://math.stackexchange.com/questions/4202911`

*Thank You for listening!*