

Institute for Advancing Intelligence (IAI), TCG CREST
End-Semestral Examination
Ph.D Program Session: 2021–2022
Discrete Mathematics

Date: 18.01.2021

Marks: 100

Time: 5 Hours

Answer as much as you can. Total marks is 120 and the maximum you can score is 100. Copying is strictly prohibited.

1. Tower of Hanoi (ToH) is a mathematical puzzle where we have three rods and n disks namely d_1, \dots, d_n (increasing order of size). The objective of the puzzle is to move the entire stack from rod 1 to rod 3, obeying the following simple rules:
 - Only one disk can be moved at a time.
 - Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack i.e. a disk can only be moved if it is the uppermost disk on a stack.
 - No disk may be placed on top of a smaller disk.

Now consider the following problems and find the solution:

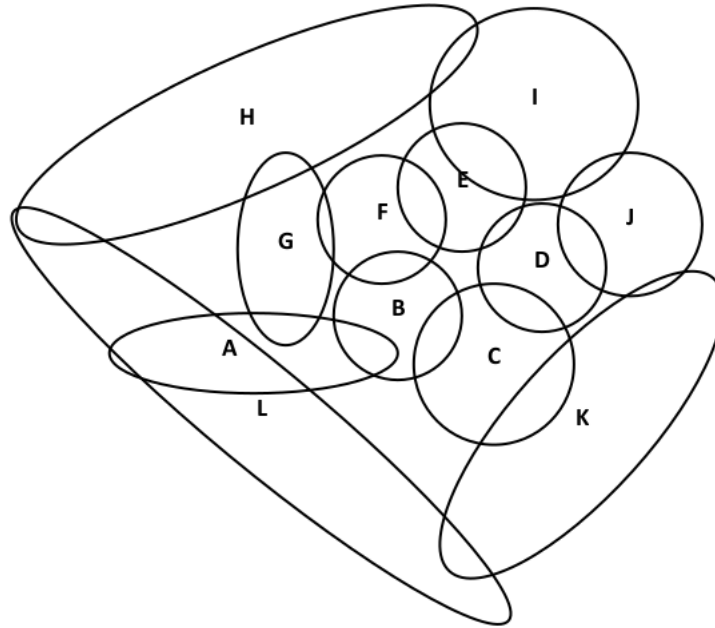
- (a) Assume that moving disk d_i from one rod to another takes i unit time. How much time is required to solve the problem?
- (b) Suppose you are not allowed to move any disk directly from rod 1 to rod 3, and vice versa (i.e. every move should involve rod 2). So, if you start with a single disk, it will take 2 moves (rod 1 to rod 2, then rod 2 to rod 3). Find the minimum number of moves required for $n = 2$, and then try to find an expression for the minimum number of moves required for general n .

[6+9=15]

2. (a) Suppose Saikat would like to send some information using a sequence of binary digits, 0 and 1. Assume sending the bit 0 takes 1 unit of time, while sending the bit 1 takes 2 units of time. Here the size of a message will be defined as the number of units time it takes. How many different messages can Saikat send in n time units?

- (b) The private nursing home Bpollo is charging INR 780 for the Covid vaccine and you have to pay the exact amount in a vending machine placed in the nursing home. Determine the number ways Shreya may pay the amount for both the cases (i) where the order in which the notes are given does not matter, and (ii) when the order does matter. Assume that the machine accepts notes of denomination 10, 20, 50, 100, 200, 500, 2000. **[7+8=15]**
3. (a) Prove that $\sum_{k=0}^n 3^k \binom{n}{k} = 4^n$.
- (b) Tamojit brings 30 different chocolates in the lab. Find the number of ways in which he can distribute the chocolates among his 15 friends at IAI ensuring that everyone gets at least one chocolate. [Note: Tamojit knows all of them, so he can distinguish between any two.]
- (c) Find out the number of ways in which Tamojit can distribute 20 chocolates among 10 administrative staffs? [Assume that Tamojit can not distinguish the administrative staffs and he distributes the chocolates without any restriction.] **[3+6+6=15]**
4. (a) Find the number of spanning trees of K_{10} .
- (b) Show that, for each value of n , the graph associated with the alcohol $C_nH_{2n+1}OH$ is a tree. Use the fact that each carbon, oxygen, hydrogen atom must be connected to exactly 4, 2, 1 many atoms respectively.
- (c) Suppose we have an undirected graph with weights that can be either positive or negative. Do Prim's and Kruskal's algorithm produce a MST for such a graph?
- (d) Design an efficient algorithm to find the longest path between two nodes in a weighted graph. Your algorithm should work for all graphs (negative weight edges are allowed). **[2+3+3+7=15]**
5. (a) Prove that, if $G = (V, E)$ is a maximal planar graph ($|V| \geq 4$), then the degree of every vertex of G is at least 3.
- (b) Use the above result to show that every simple planar graph G with $|V| \geq 4$ has at least four vertices of degree less than 6.
- (c) Let G be a simple graph with 19 edges, and degree of each vertex is greater than 3. Knowing nothing else about G , find (i) the maximum number of vertices that G could have, (ii) the maximum number of vertices that G could have for which one can conclude whether G is planar or not. **[4+5+6=15]**

6. (a) How many colors do you need to edge color a bi-partite graph?
- (b) There are 12 radio stations in India broadcasting music in local languages: A, B, C, D, E, F, G, H, I, J, K and L. Each radio station has a specific area in which people can listen to it, as drawn below. Unfortunately, some areas overlap. The officers



of All India Radio can license specific wave frequency to each station. But, they want to use the minimal amount of different frequencies, in order to minimize costs. Nevertheless, in an area of overlap, each radio station has to have different frequency; otherwise the listeners would not be able to listen to their station. Formulate the problem as a graph theoretic problem and find how to distribute the frequencies amongst the radio stations so that minimal amount of different frequencies is used? **[3+12=15]**

7. (a) Two married couples want to cross a river. They can only use a boat that can carry one or two people from one shore to the other shore. Each husband is extremely jealous and is not willing to leave his wife with the other husband, either in the boat or on shore. How can these four people reach the opposite shore?
- (b) Given the 5×5 checkerboard with a few blocks removed (marked with \times), how can you figure out whether it is possible to tile the

rest of the board with dominoes (a domino being two adjacent squares)? Use your approach to solve the generalized version of the problem i.e., given an $m \times n$ checkerboard with a few blocks removed, can you tile it with dominoes? **[6+9=15]**

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8. Consider an inter-IAI chess competition where each plays exactly once with everyone else. Assume each match has a winner, tie-breakers are used for matches that results in a tie. A player T is called champion if for all the other players S , (i) either T beats S , or (ii) there exists another player R such that T beats R and R beats S .
- Is it possible to have Sagnik as the unique champion?
 - Suppose Jayanta wins only a single match. Is it possible that Jayanta becomes one of the champions of the tournament?
 - Show that it is possible that both Sangita and Shital becomes champion.
 - Show that it is impossible to have Sangita and Shital as the *only* champions. **[2+3+3+7=15]**