ABSTRACTION IN MATHEMATICS

1. Introduction

What does it mean to say that mathematics is abstract?

Mathematics is a self-contained system separated from the physical and social world:

- Mathematics uses everyday words, but their meaning is defined precisely in relation to other mathematical terms and not by their everyday meaning. Even the syntax of mathematical argument is different from the syntax of everyday language and is again quite precisely defined.
- Mathematics contains objects which are unique to itself. For example, although everyday language occasionally uses symbols like x and P, objects like x^0 and $\sqrt{-1}$ are unknown outside mathematics.
- A large part of mathematics consists of rules for operating on mathematical objects and relationships. It is important that students learn to manipulate symbols using these rules and no others.

The essence of abstraction in mathematics is that mathematics is self-contained: An abstract mathematical object takes its meaning only from the system within which it is defined.

Historically, mathematics has seen an increasing use of axiomatic methods, especially over the last two centuries. For example, numbers were initially mathematical objects based on the empirical idea of quantity. Then mathematicians such as Dedekind and Peano reconceptualized numbers in axiom systems which were independent of the idea of quantity. Euclid, Hilbert, and others performed a similar task for geometry. But, as Kleiner (1991) states, whereas Euclid's axioms are idealizations of a concrete physical reality in the modern view axioms are simply assumptions about the relations among the undefined terms of the axiomatic system. In other words, mathematics has become increasingly independent of experience, therefore more self-contained and hence more abstract.

To emphazise the special meaning of abstraction in mathematics, we may say that mathematical objects are abstract-apart. Their meanings are defined within the world of mathematics, and they exist quite apart from any external reference.

So why is mathematics so useful?

Mathematics is used in predicting and controlling real objects and events, from calculating a shopping bill to sending rockets to Mars. How can an abstract-apart science be so practically useful?

One aspect of the usefulness of mathematics is the facility with which calculations can be made: You do not need to exchange coins to calculate your shopping bill, and you can simulate a rocket journey without ever firing one. Increasingly powerful mathematical theories (not to mention the computer) have led to steady gains in efficiency and reliability.

But calculation facility would be useless if the results did not predict reality. Predictions are successful to the extent that mathematics models appropriate aspects of reality, and whether they are appropriate can be validated by experience. In fact, one can go further and claim that the mathematics we know today has been developed (in preference to any other that might be imaginable) because it does model significant aspects of reality faithfully.

How is it that the axiomatic method has been so successful in this way? The answer is, in large part, because the axioms do indeed capture meaningful and correct patterns. For instance, Euclid's axioms try to capture properties of geometric patterns or objects like *triangles, circles, parabola etc* that we encounter in real life. These geometric objects are built from two basic objects *viz* **points** and **lines**. The axioms are about these basic geometric objects and from these axioms we derive *theorems* which are essentially properties of these geometric objects. Thus, as Kliener has said, Euclid's axioms are "idealizations of a concrete physical reality".

Many fundamental mathematical objects (especially the more elementary ones, such as numbers and their operations) clearly model reality. Later developments (such as Combinatorics and differential equations) are built on these fundamental ideas and so also reflect reality–even if indirectly. Hence all mathematics has some link back to reality.

EMPIRICAL ABSTRACTION IN MATHEMATICS LEARNING

Students learn about many fundamental, abstract mathematical objects in school. In what follows, we discuss the meaning of abstraction in this learning context. We begin by looking at some examples.

Addition: Between the ages of 3 and 6, most children learn that a given set of objects contains a fixed number of objects. A little later, they realize that two sets can be combined and that the number of objects in the combined set can be determined from the number of objects in each set–a procedure which later becomes the operation of addition. Students learn these fundamental arithmetical ideas from counting experiences: They find that repeatedly counting a given set of objects always gives the same number, no matter how often it is done and in which order. As they recognize more and more patterns, counting a combined set is gradually replaced by counting on.

Angles: There is good evidence that, at the beginning of elementary school, students already have knowledge of corners, slopes, and turns. To acquire a general concept of angle, students need to see the similarities between them and identify their essential common features (two lines meeting at a point, with some significance to their angular deviation). Even secondary students find it difficult to identify angles in slopes and turns, where one or both arms of the angle have to be imagined or remembered.

Rate of change: The most fundamental idea in calculus is rate of change, leading to differentiation. A major reform movement over the last decade or so has been concerned with making this idea more meaningful by initially exploring a range of realistic rate of change situations. In this way, students build up an intuitive idea of rate of change before studying the topic abstractly.

Characteristics of empirical abstraction

The above examples show how fundamental mathematical ideas are based on the investigation of real world situations and the identification of their key common features. Hence, a characteristic of the learning of fundamental mathematical ideas is similarity recognition. The similarity is not in terms of superficial appearances but in underlying structure for example, in counting, space, and relationships.

This process of similarity recognition followed by embodiment of the similarity in a new idea is an empirical abstraction process.

Thus, abstracting is an activity by which we become aware of similarities among our experiences. Classifying means collecting together our experiences on the basis of these similarities. An abstraction is some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having the similarities of an already formed class, and to distinguish between abstracting as an activity and abstraction as its end-product.

Thus number, addition, angle and rate of change are all empirical concepts, and they take their place in students' learning alongside other empirical concepts such as colour, texure, and taste.

There is a distinction between abstraction on the basis of superficial characteristics of physical objects and abstraction on the basis of relationships perceived when the learner manipulates these objects. But both are based on our physical and social experience, and in both similarity recognition is essential.

FROM EMPIRICAL CONCEPT TO MATHEMATICAL OBJECT

When students learn a fundamental mathematical idea in the way described above, three things happen: They learn an empirical concept, they learn about a mathematical object, and they learn about the relationship between the empirical concept and the mathematical object. Empirical concepts are often rather fuzzy and difficult to define. For example, the empirical concept of circle is that of a perfectly round object–but perfect roundness can only be defined by showing examples. A circle becomes a mathematical object only when it is defined as the locus of points equidistant from a fixed point: It is then clearly defined in terms of other mathematical objects. However, for this definition to be meaningful, an individual must see that the locus of points equidistant from a fixed point gives a perfectly round object and vice versa.

Abstraction in mathematics or in other disciplines is best understood by axiomatic treatment. It helps us to obtain proofs of statement most economically.

But why do we need a proof?

Proofs are the guts of mathematics. Producing a proof of a statement is the basic methodology whereby we can ascertain that the statement is true. Anyone who wants to know what mathematics is about must therefore learn how to write down a proof or at least understand what a proof is. The sciences also use the same methodology to deduce complex phenomena from first principles. Thus all who want to study science would benefit from learning about proofs as well.

In a larger context, if anyone has any wish at all to find out how human beings can rationally distinguish in an empirical sense, right from wrong or true from false, he or she would find in mathematical proofs the purest form of how this is done.

One may ask why not put most or all of our weight on experimentally verifying a statement rather than the theorem–proof aspect of verifying it. Perhaps the argument is that because hands-on experiments are as efficient at arriving at the truth as abstract arguments, why not bypass this arduous task of writing down proofs altogether? In case such a statement does not immediately appear as being silly, let us try to convince ourselves with a simple example.

A standard problem in number theory is to find integer solutions to equations of the following type (the Fermat-Pell equation):

$$x^2 - 1141y^2 = 1.$$

This is of course the same as looking for positive integers y so that $1 + 1141y^2$ is a perfect square. (We exclude the obvious solutions: $x = 1$, $y = 0$.) This is a problem tailormade for experimentation on the calculator. Starting with $y = 1, 2, 3, \ldots$ we can work our way up. The case $y = 1$ is no good because $1,142$ is not a perfect square, for the simple reason that the square of any number must end in $1, 4, 5, 6$, or 9. In fact, nothing works up to 100. For example, with $y = 99$, we get

$$1 + 1141(99)^2 = 11,182,942$$

so that for the same reason it is not a perfect square. Similarly, nothing works up to $y = 100,000$. For example, with $y = 23,456$, we get

$$1 + 1141(23,456)^2 = 627,759,870,977,$$

and because it ends in 7, it is not a square. If you try $y = 45,678$, then

$$1 + 1141(45,678)^2 = 2,380,673,319,445,$$

and

$$(1,542,9432)^2 = 2,380,673,101,249 < 2,380,673,319,445$$
$$< 2,380,676,187,136 = (1,542,9442)^2.$$

In fact, for all integers y all the way up to 10^{25}, $1+1141y^2$ is never a perfect square. In terms of experimentation, one would have given up long before this and concluded that this particular Fermat-Pell equation has no integer solutions in x and y. But in fact, we can prove that there are an infinite number of integer pairs x and y that satisfy this equation, the smallest being:

$$y = 30,693,385,322,765,657,197,397,208$$

and

$$x = 1,036,782,394,157,223,963,237,125,215.$$

The above illustration is not to belittle the importance of experimentation, because experimentation is essential in science; rather, it is just to emphasize that Mathematics is concerned with statements that are true, forever and without exceptions, and there is no other way of arriving at such statements except through the construction of proofs.

What is the axiomatic method?

The axiomatic method is quite straight-forward: –Make some assumptions (axioms) that we cannot prove, but seems plausible e.g. Euclid's axioms.

We then use these axioms to deduce or prove, logically, some statements which are called theorems. Thus theorems are logical consequences of some of the axioms. Once a theorem is proven it can be used in subsequent proofs.

The modern notion of the axiomatic method developed as a part of the conceptualization of mathematics starting in the nineteenth century. The basic idea of the method is the capture of a class of structures as the models of an axiomatic system. What condition does an axiom system have to satisfy?

Obviously, the following requirements must be satisfied:

(1) The models of the axiomatic theory have to comprise all and only the intended structures.
(2) All theorems are logical consequences of the axioms.
(3) The derivation of theorems from axioms must not introduce any new information into the conclusion.
(4) The logic used must be complete.

The first exposure of an axiomatic treatment that one encounters is the Eucledian Geometry. Euclid's works are the first still-existing record of an attempt to apply the axiomatic method.

Euclid wrote down a number of definitions:

- A point is that which has no part.
- A line is breadthless width.
- The ends of a line segment are points.

and twenty others such definitions.

Euclid also wrote some common notions, which today we describe as axioms:

- Things which are equal to same thing are also equal to each other. In other words, if $a = c$ and $b = c$, then $a = b$.

- If equals are added to equals, then the wholes are equal. In other words, if $a = b$ and $c = d$, then $a + c = b + d$.
- Similarly, if equals are subtracted from equals, then the remainders are also equal.
- Things which coincide with each other are equal.
- The whole is greater than the part.

Euclid then introduced five axioms (he called them postulates):

- A straight line segment can be drawn joining any two points.
- Any straight line segment can be extended indefinitely in a straight line .
- Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- All right angles are congruent.
- If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate.

From these axioms he went on to deduce hundreds of propositions– what we now call theorems.

The next axiomatic treatment that you have seen in your first course in real analysis is the introduction of real number system as a complete ordered field.

Thus, given a set of axioms in some specific discipline, we may go forward and prove as many theorems as possible. Once a theorem is proved, it can be used in the proof of a subsequent statement. What is really nice is that if we have another system that satisfy the same axioms, then all the theorems must also be valid in the new system.

During the course we will briefly touch upon the following examples of axiomatic treatment.

(1) From naive set theory to ZFC axioms of Set Theory.
(2) Category Theory– a unified abstract approach to study various branches of mathematics. A brief introduction to Monoidal category.

Reference: Kleiner, I. (1991); Rigor and proof in mathematics: A historical perspective. Mathematics Magazine, 64, $291 - 314$.