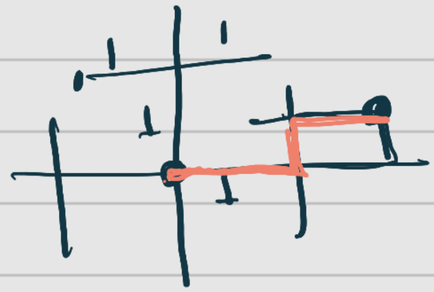
Coarse Geometry.

Objects of study: metric spaces.

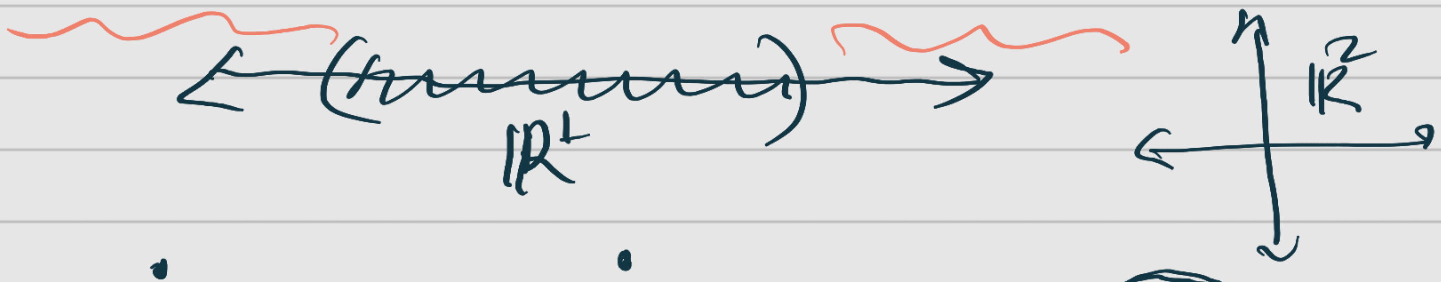
(X, d)

Examples: Euclidean space $(\mathbb{R}^n)$

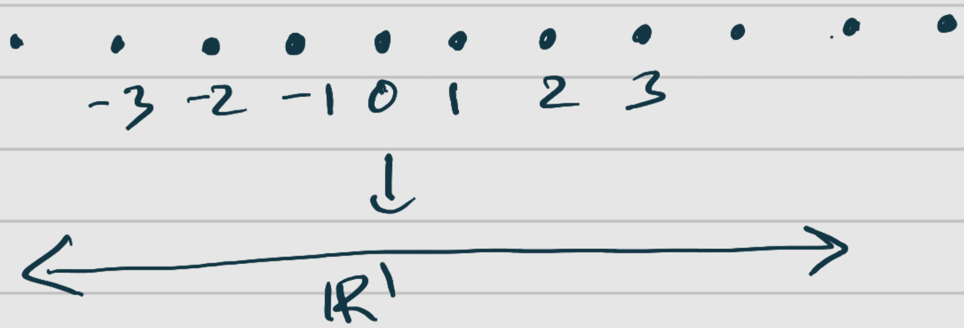


Any graph can be given a metric.

Coarse geometry is the study of the "geometry at infinity" of the space.



The study of structure that survives as one zooms infinitely far away from the space.



$\mathbb{Z}$ is coarsely same as $\mathbb{R}$.

Why care: We do that all the

...ly care: We do that all the time. (see the pictures at the end of this note)

Geometric group theory:

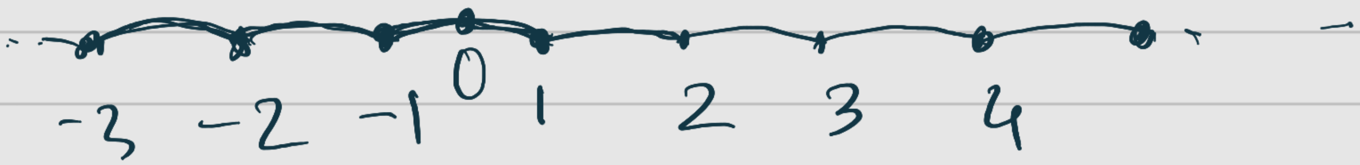
Gromov 83': G, finitely generated by some set S.

$\text{Cayley}_S(G)$ is a graph.

vertex set $\equiv$ {elements of G}.

edge set $=$ $g_1, g_2 \in G$ is ~~connect~~ connected by an edge if $g_1 = g_2 s$ for some $s \in S$

$\mathbb{Z} = \langle 1 \rangle$



Thm (Milnor-Švarc) for any two generating sets S_1 and S_2 of G, $\text{Cayley}_{S_1}(G)$ and $\text{Cayley}_{S_2}(G)$ are coarsely same.

Algebraic properties of the group G

$\updownarrow$

coarse geometry of $\text{Cay}(G)$

Definition: (Coarse embedding).

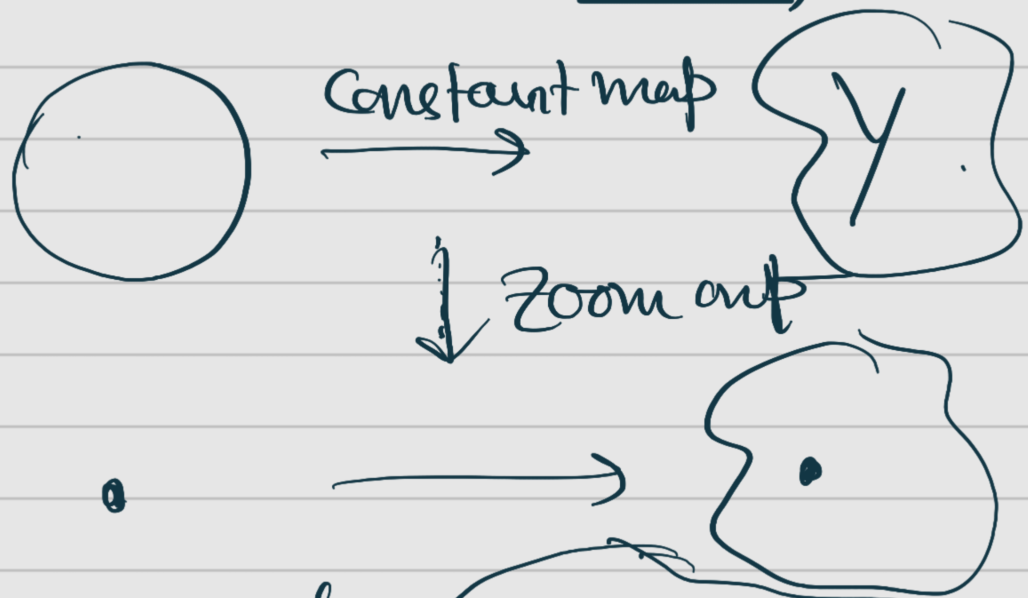
$f : X \to Y$ is called a coarse embedding if there exist two nondecreasing proper $\rho_+, \rho_- : [0,\infty) \to [0,\infty)$

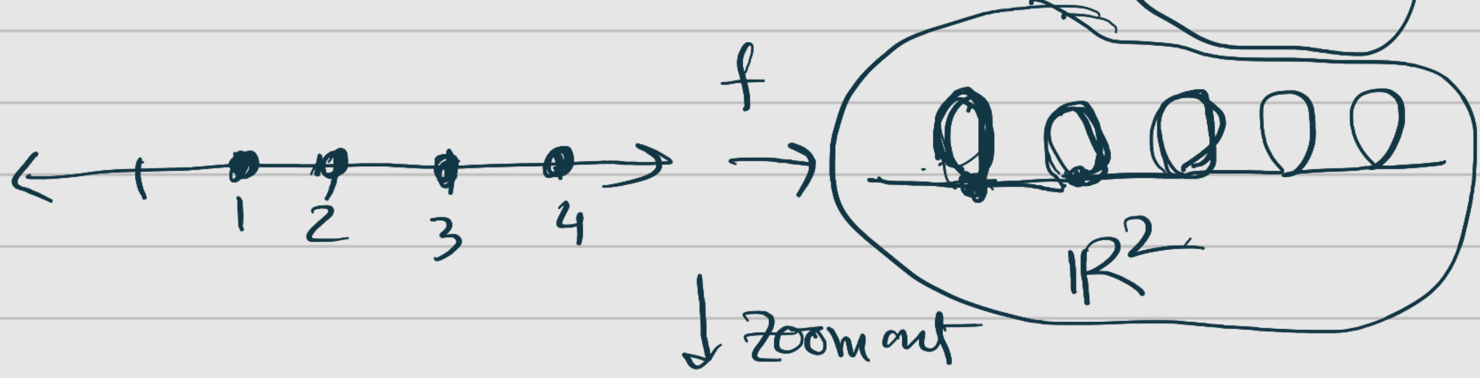
$$\rho_-(d(x,y)) \leq d(f(x),f(y)) \leq \rho_+(d(x,y))$$

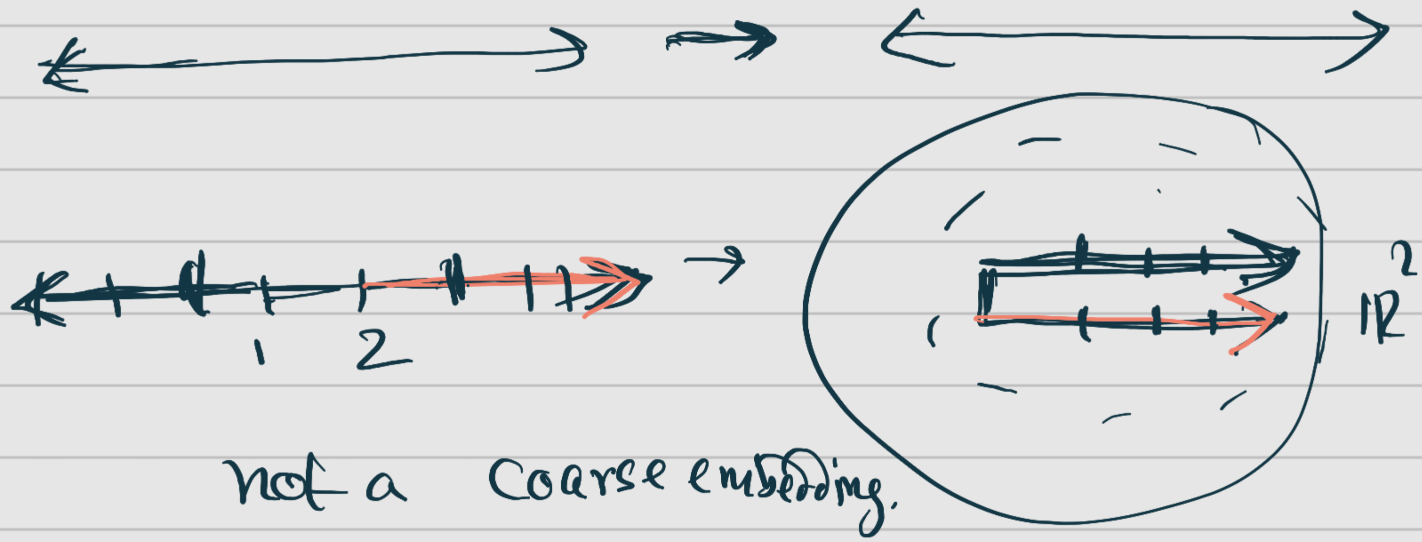
for all $x, y \in X$.

- If $\rho_+, \rho_1 = id$ then $d(f(x),f(y)) = d(x,y)$

Example:







not a coarse embedding.

Novikov Conjecture:

(a) Smooth invariants are homotopy

some smooth structures and [illegible]
invariants.

(Yu 98'). If $\pi_1(M^n)$ coarsely embed inside a ~~Banach~~ Hilbert space.
then M^n satisfies Novikov Conjecture.
where M^n is closed aspherical manifold.

Q: What are the obstruction to embed a group into a Hilbert space.

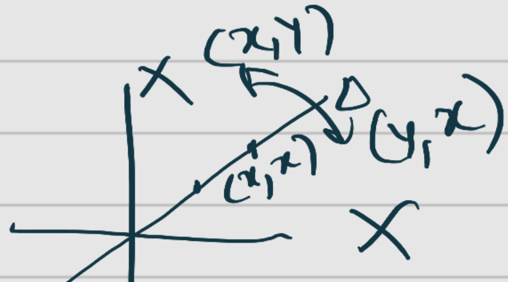
$\boxed{\text{expanders.}}$: graphs that are <u>sparse but highly connected</u>

existence of expanders in G.
obstruct ~~to~~ coarse embedding into any ~~Banach~~ Hilbert space.

Q. Given any two space X and Y
How to obstruct <u>coarse embedding</u> X into Y.

<u>Thm</u> (Van Kampen). Given a space X.

there is a class $\alpha \in \underline{H^{k}_{\mathbb{Z}_2}(X \times X - \Delta)}$



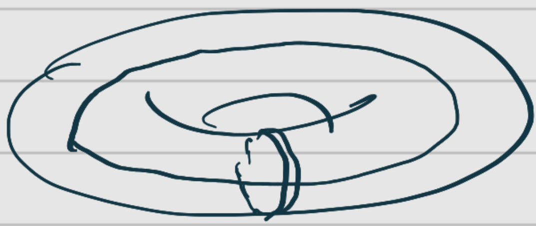
such that $\alpha \neq 0$.
then $X \not\hookrightarrow \underline{\mathbb{R}^k}$.

embedding.

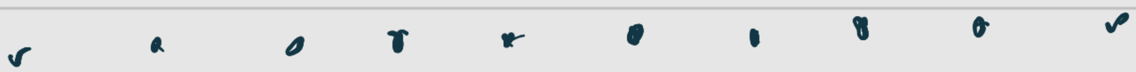
Is there any analogous statement in the coarse setting.

Thm (B). Given a space X, there is a class $\alpha \in HX^k_{\mathbb{Z}_2}(X \times X - \Delta)$ such that $\alpha \neq 0 \Rightarrow X \underset{\text{coarsely embed}}{\not\hookrightarrow} \mathbb{R}^{k-1}$

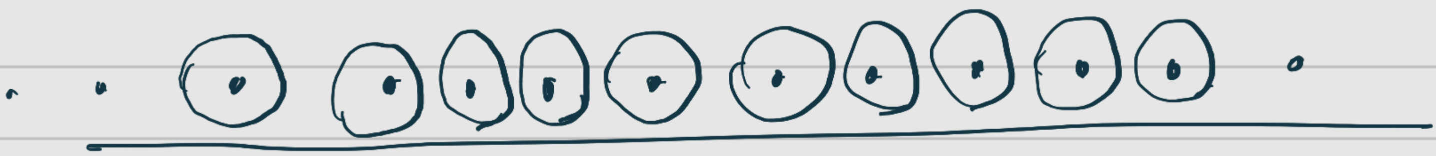
Coarse cohomology (Roe)

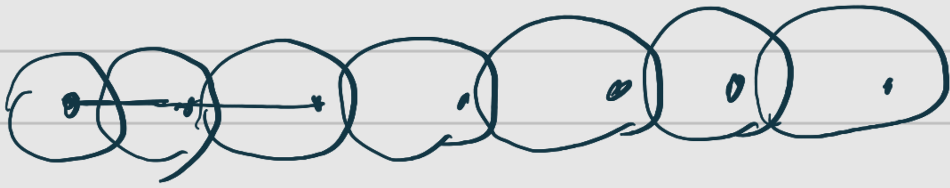


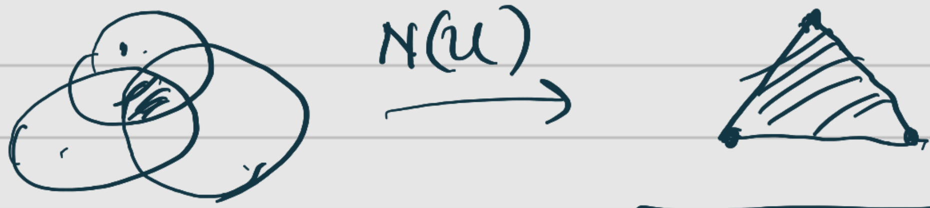
Coarse objects can be discrete.



coarsely



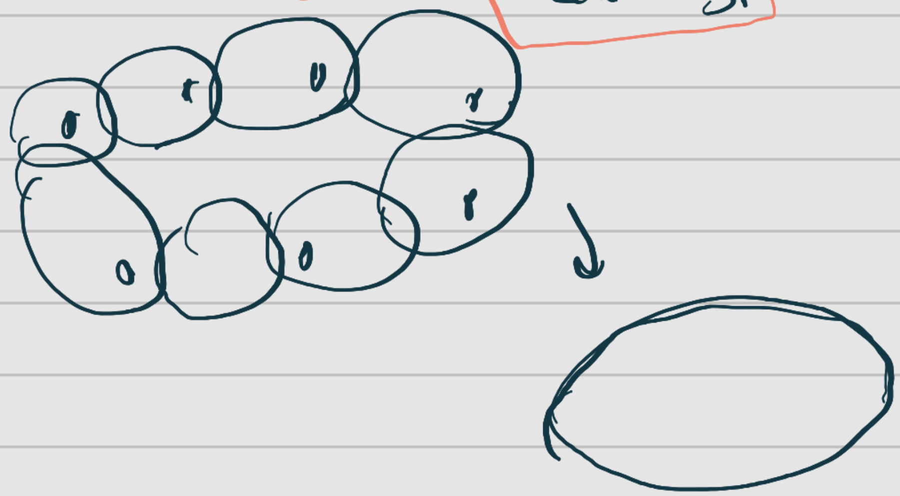




$\{\mathcal{U}_n\} \longrightarrow \boxed{N(\mathcal{U}_n)}.$

Coarse cohomology $\approx \boxed{\varprojlim H^k_c N(\mathcal{U}_n)}$ *

Compactly supported cohomology



$N(\mathcal{U}_1) \longrightarrow N(\mathcal{U}_2) \longrightarrow N(\mathcal{U}_3)$

$H^k_c(N(\mathcal{U}_1)) \leftarrow H^k_c(N(\mathcal{U}_2)) \leftarrow$

$C^k_c(N(\mathcal{U}_1)) \leftarrow C^k_c(N(\mathcal{U}_2)) \leftarrow C^k_c(N(\mathcal{U}_3)) \cdots$

$\boxed{H\left(\varprojlim C^k_c(N(\mathcal{U}_n))\right)}$ *

<u>Partial converse of van Kampen's theorem:</u>

Let X be a simplicial complex with $\dim X = n$

and $n \neq 2$. Then $\exists\, \alpha \in H^{2n}_{\mathbb{Z}_2}(X \times X - \Delta)$ such that, $\alpha \neq 0 \iff X \not\hookrightarrow \mathbb{R}^{2n}$.

$$\boxed{\{\varphi \in C^n(X) \mid \ ? \ \}}$$

The above pictures does not make much sense if you zoom in (try it!!). But when seen from far, they become meaningful.

In some sense, their large-scale structure is more important than their small-scale details.