A Category $\mathcal{C}$ consists of a collection of objects, denoted by $Ob(\mathcal{C})$ and for each pair of objects $X, Y \in Ob(\mathcal{C})$ a set $hom_{\mathcal{C}}(X, Y)$ s.t for each object $X \in Ob(\mathcal{C})$ $\exists$ a distinguished morphism $1_X \in hom_{\mathcal{C}}(X, X)$, called the identity morphism and for objects $X, Y, Z \in Ob(\mathcal{C})$, $\exists$ a composition law $hom_{\mathcal{C}}(Y, Z) \times hom_{\mathcal{C}}(X, Y) \longrightarrow hom_{\mathcal{C}}(X, Z)$ which satisfies the following axioms:

i) (identity) For $1_X$, and $1_Y$ and a morphism $f \in \mathrm{hom}_{\mathcal{C}}(X, Y)$

hTop

$$1_Y \circ f = f = f \circ 1_X$$

(here '$\circ$' is the composition law).

We denote a morphism

$$f \in \mathrm{hom}_{\mathcal{C}}(X, Y)$$

by $f : X \longrightarrow Y$

A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is a category $\mathcal{C}'$ s.t

$$\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$$

and $$\hom_{\mathcal{C}'}(X', Y') \subset \hom_{\mathcal{C}}(X', Y')$$

$$X', Y' \in \mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$$

<u>Example</u> (i) Let $\mathcal{C} = Grp$ — the category of groups

$\mathcal{S} = Set$ — the category of sets.

$Grp \subset \mathcal{S}$

Let $\mathcal{C} = Grp$ — category of groups

and $\mathcal{C}' = Ab$ — category of abelian groups.

Let $G, H$ be abelian groups $\hom_{Ab}(G, H) = \hom_{Grp}(G, H)$

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a full-subcategory if for any objects $X, Y \in Ob(\mathcal{C}')$

$$\hom_{\mathcal{C}'}(X, Y) = \hom_{\mathcal{C}}(X, Y).$$

A category $\mathcal{C}$ is called <u>small</u> if $Ob(\mathcal{C})$ is a set.

Example Let $X$ be a topological space. Let $x, y \in X$. A path in $X$ from $x$ to $y$ is a continuous function $\alpha : [0,1] \longrightarrow X$ s.t. $\alpha(0) = x$
$\alpha(1) = y$


Let $\beta : [0,1] \longrightarrow X$ be another path from $x$ to $y$.

We say that "$\alpha$ is homotopic to $\beta$ relative endpoints" written as $\alpha \simeq \beta$ rel $(0,1)$

if $\exists$ a continuous function

$$F : I \times I \longrightarrow X, \qquad I = [0,1]$$

Other notation:

For each $t \in [0,1]$ let $\alpha_t : [0,1] \to X$ be defined by $\alpha_t(s) = F(s,t)$

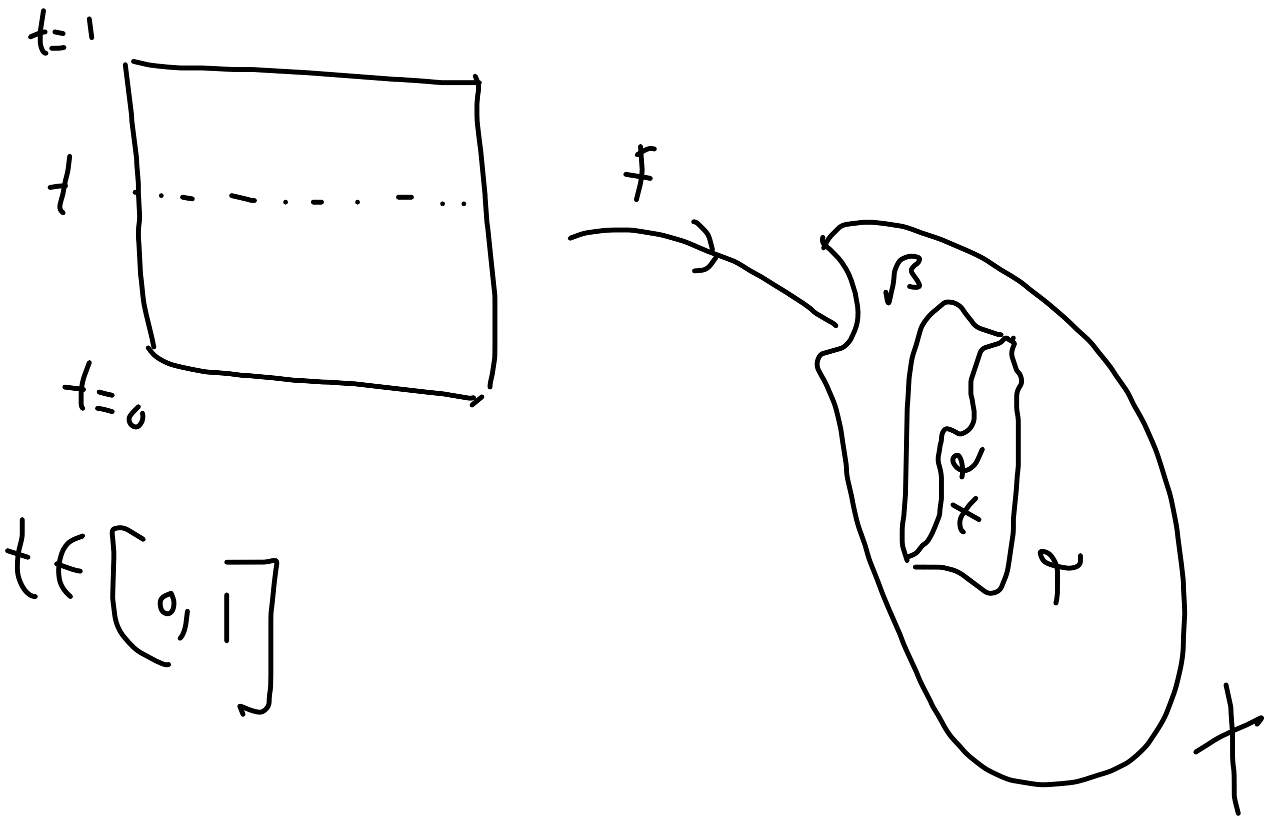
s.t.

$$F(s,0) = \alpha(s)$$

$$F(s,1) = \beta(s)$$

$$F(0,t) = x$$

$$F(1,t) = y \quad \forall t \in [0,1]$$



Let $P(X; x, y)$ be the set of all paths in $X$ from $x$ to $y$

Consider $P(X; x, y)$ as a topological space

Then Continuity (joint) of $F : I \times I \to X$ will ensure that the assignment

$$\begin{aligned} I &\longrightarrow P(X; x, y) \\ t &\longmapsto f_t \end{aligned}$$

is continuous.

First note that "$\simeq$ rel $(0,1)$"

is an equivalence relation on $P(X; x, y)$.

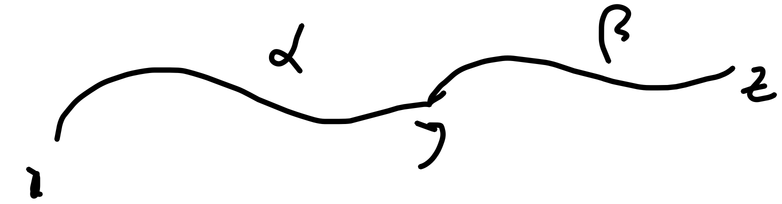
$[\alpha]$ — equivalence class of $\alpha$

Define a category $\Pi X$ Given a space $X$; whose objects are points of $X$

Given two points $x, y \in X$ $(x, y \in Ob(\Pi X))$

$$\hom_{\Pi X}(x,y) = \{ [\alpha] \mid \text{homotopy classes of paths from } x \text{ to } y \}.$$

If $x, y, z \in X$




and $\alpha : x \to y$, $\beta : y \to z$ are paths.

Then $\alpha * \beta : x \to z$

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

$[\alpha] \cdot [\beta] = [\alpha * \beta]$ — Check that this is well-defined.

$$[\alpha] = \left\{ \alpha' : [0,1] \to X \;\middle|\; \begin{array}{l} \alpha'(0) = x \text{ and } \alpha' \simeq \alpha \text{ rel } \{0,1\} \\ \alpha'(1) = y \end{array} \right\}$$

Fundamental groupoid.

Small Category.

$\alpha : x \to y$

$[\alpha]^{-1} := [\alpha^{-1}]$

$\alpha^{-1} : y \to x$

$\alpha^{-1}(t) = \alpha(1-t)$

To Check This is well defined

verify :

if $\alpha \simeq \alpha'$ rel$(0,1)$, $\alpha, \alpha' : x \to y$

and $\beta \simeq \beta'$ rel$(0,1)$, $\beta, \beta' : y \to z$

Then $\alpha * \beta \simeq \alpha' * \beta'$ rel$(0,1)$.

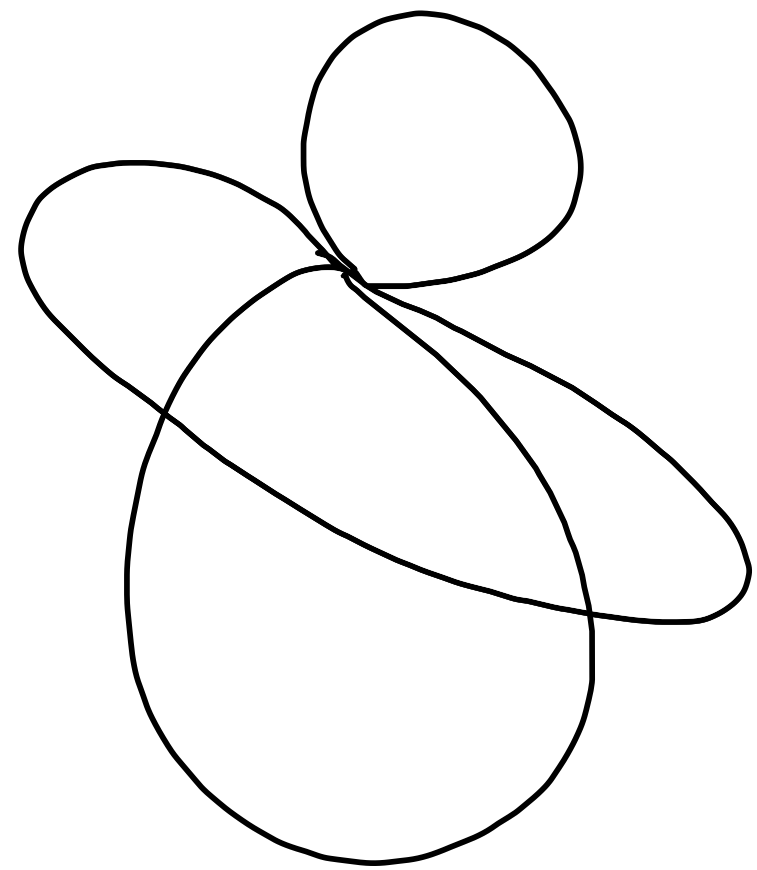
$1_x = [c_x]$, $c_x : I = [0,1] : \longrightarrow X$

$c_x(s) = x \;\; \forall s$.

Take $x = y$

$$P(X; x, x)$$

A path from $x$ to $x$ is a loop.



$$[X; x, x] := P(X; x, x) \Big/ \simeq \text{rel}\{0,1\}$$

$= \pi_1(X, x)$ — grp — called the fundamental grp of $X$.

<u>Functor</u>: Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

A covariant functor $F: \mathcal{C} \to \mathcal{D}$ is an assignment which assigns to each object $X \in ob(\mathcal{C})$, an object $F(X) \in ob(\mathcal{D})$ and to each morphism $f: X \to Y$ in $hom_{\mathcal{C}}(X, Y)$

a morphism

$$F(f) : F(X) \longrightarrow F(Y)$$

Similarly Contravariant functor is defined.

and the following rule must be satisfied

i) $F(1_X) = 1_{F(X)}$

ii) $F(g \circ f) = F(g) \circ F(f)$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] \arrow[d, "F"'] & Y \arrow[r, "g"] \arrow[d, "F"] & Z \\
F(X) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g) \circ F(f)"'] & F(Y) \arrow[r, "F(g)"] & F(Z)
\end{tikzcd}$$

i) Forgetful functor.

$$\mathcal{F} : \mathrm{Top} \longrightarrow \mathrm{Set}.$$

$$\mathrm{Grp} \longrightarrow \mathrm{Set}$$

$$\mathrm{Vect}_k \longrightarrow \mathrm{Vect}_k.$$

$$V \longrightarrow V^* = \mathrm{Hom}(V, k)$$

$$f : V \to W \longmapsto f^* : W^* \to V^*$$

ii) $\pi_1 : \mathrm{Top}^* \longrightarrow \mathrm{Grp}$

$$(X, x) \longrightarrow \pi_1(X, x).$$

$$H_k : \mathrm{Top} \to \mathrm{Ab}.$$

$$X \to H_k(X)$$

Let $F, G : \mathcal{C} \to \mathcal{D}$ be covariant functors

A natural transformation

$$\eta : F \to G$$

is an assignment

$$X \in Ob(\mathcal{C}) \longrightarrow \eta(X) : F(X) \to G(X)$$

$$f \downarrow \in hom_{\mathcal{C}}(X, Y) \longrightarrow F(f) \downarrow \circlearrowright \downarrow G(f)$$

$$Y \in Ob(\mathcal{C}) \qquad \eta(Y) : F(Y) \to G(Y)$$

commutative.

If $\eta(X)$ is an isomorphism $\forall X$ then $\eta$ is called an equivalence.