Category Theory

A Category $\mathcal{C}$ Consists of

- a Collection of objects, denoted by $Ob(\mathcal{C})$
- For each pair of objects $X, Y \in Ob(\mathcal{C})$ $\exists$ a set $hom(X,Y)$ (This may be empty) elements of $hom(X,Y)$ are called morphisms and denoted by $f: X \to Y$ s.t.
- For each object $X \in Ob(\mathcal{C})$, $\exists$ a morphism $1_X \in hom(X,X)$
- identity morphism and

for objects $X, Y, Z \in Ob(\mathcal{C})$

$\exists$ a composition law

$$\begin{aligned} \hom(Y,Z) \times \hom(X,Y) &\longrightarrow \hom(X,Z) \\ (g,f) &\longmapsto g \circ f \end{aligned}$$

This composition law satisfies

(i) Identity condition: for $1_X, 1_Y$ and $f: X \to Y$

$$1_Y \circ f = f = f \circ 1_X$$

(ii) associativity: Given $f: X \to Y$, $g: Y \to Z$, and $h: \; \to W$ we must have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Examples: (1) $\mathcal{C} = \mathbf{Set}$

Objects are sets

Given sets X, Y, a morphism $X \to Y$ is a set function

Composition law is the usual composition of functions

1_X — identity function

(ii) $\mathcal{C} = \mathbf{Grp}$ - Category of groups

Objects are groups. Given groups G, H a morphism $f: G \to H$ is a group homomorphism.

(iii) $\mathcal{C} = \mathrm{Top}$.

Objects are topological spaces.

Given topological spaces X, Y, a morphism $f : X \to Y$ is a continuous function.

(iv) $\mathcal{C} = \mathrm{Top}^*$ Category of pointed topological spaces.

Objects are topological spaces along with a preferred base point. (X, x_0) $x_0 \in X$.

Given pointed spaces (X, x_0) (Y, y_0) a map $f : (X, x_0) \to (Y, y_0)$ is a continuous function $f : X \to Y$ s.t. $f(x_0) = y_0$.

v) Let k be a field. Then $\mathcal{C} = \mathrm{Vect}_k$

Objects are finite dimensional vector spaces over k and a morphism $f: V \to W$ is a linear map.

Defn: In a category $\mathcal{C}$, a morphism $f: X \to Y$ is called an isomorphism if $\exists$ a morphism $g: Y \to X$ s.t. $g \circ f = 1_X$ & $f \circ g = 1_Y$

vi) Let G be a given group.

Let $\mathcal{G}$ denote a category s.t. $Ob(\mathcal{G}) = \{*\}$

$\hom(*,*) = G$. $\forall$ $g \in G$, Then

$g: * \longrightarrow *$.

$$\begin{array}{ccccc} \hom(*,*) & \times & \hom(*,*) & \longrightarrow & \hom(*,*) \\ \| & & \| & & \| \\ G & \times & G & \longrightarrow & G \end{array}$$

$1_* = e$

Composition in group i.e. group multiplication.

Let $[f]$ be the equivalence class of f

$[\cdot]$

Example

$\mathcal{C} = \mathrm{hTop}$

Objects are topological spaces and a morphism $X \to Y$ is a homotopy class of a map $f: X \to Y$

$[f] = [f']$

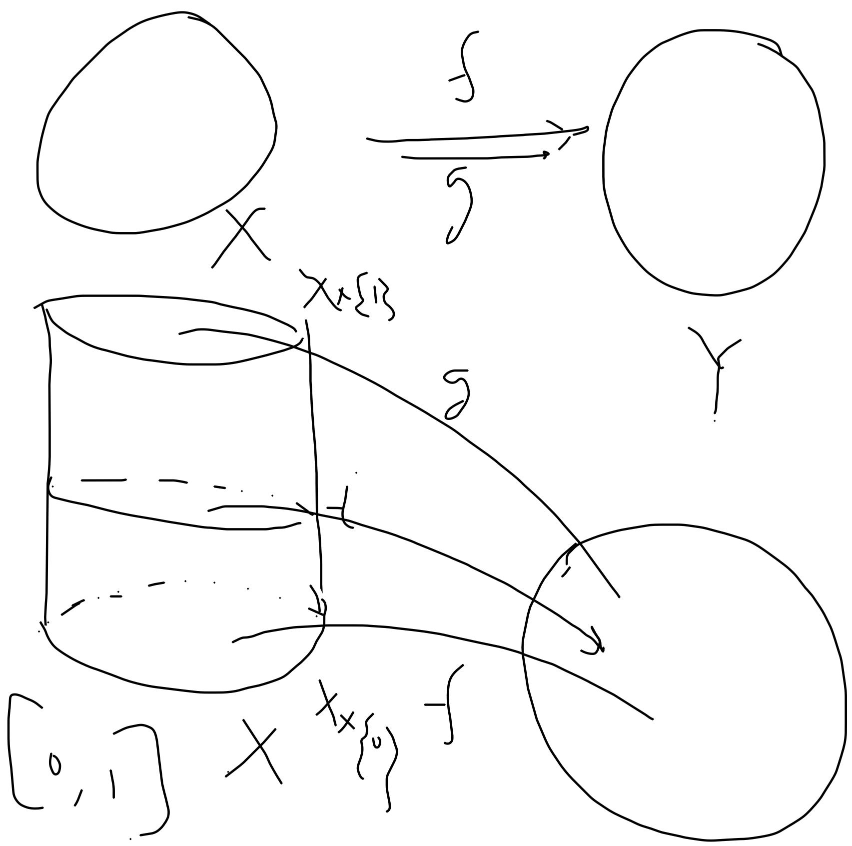
$[g] = [g']$

Let $\hom(X, Y)$ be the set of all continuous functions $X \to Y$

Check that '$\simeq$' (being homotopic) is an equivalence relation on $\hom(X, Y)$

For any $t \in [0,1]$ define $f_t : X \to Y$ by $f_t(x) = F(x, t)$

We write $f \simeq g$.



$X \times I$

$I = [0,1]$

We say f is homotopic to g if $\exists$ a continuous function $F: X \times I \to Y$ s.t. $F(x,0) = f(x)$, $F(x,1) = g(x)$ $\forall x \in X$

To define $[g] \circ [f] = [g \circ f]$

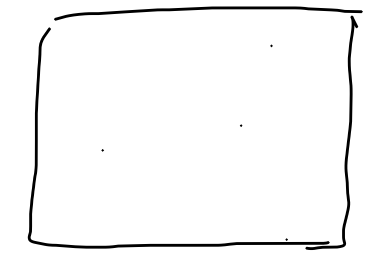
$$= [g' \circ f']$$

<u>Example</u> Let P be a partially ordered set $P = (S, \leq)$

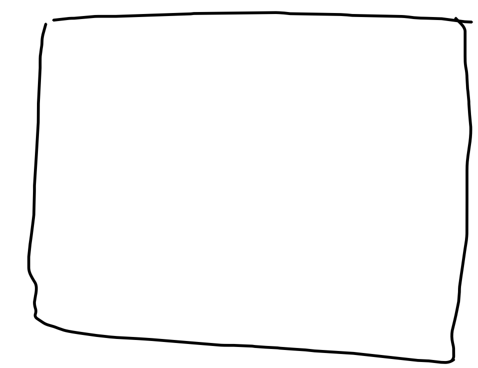
Define a Category P where objects are elements of S

Given $a, b \in S$, if a is not related to b, then define $\hom(a,b) = \emptyset$, if $a \leq b$ define $\hom(a,b) = \{a \leq b\}$

Let $I^{(2)} = I \times I$, $I = [0,\ ]$



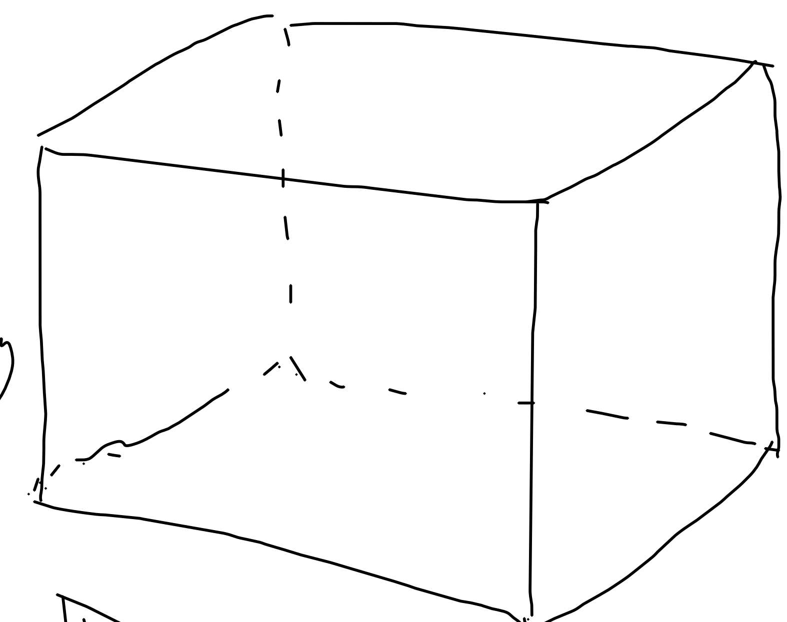
$I^{(3)} = I \times I \times I$



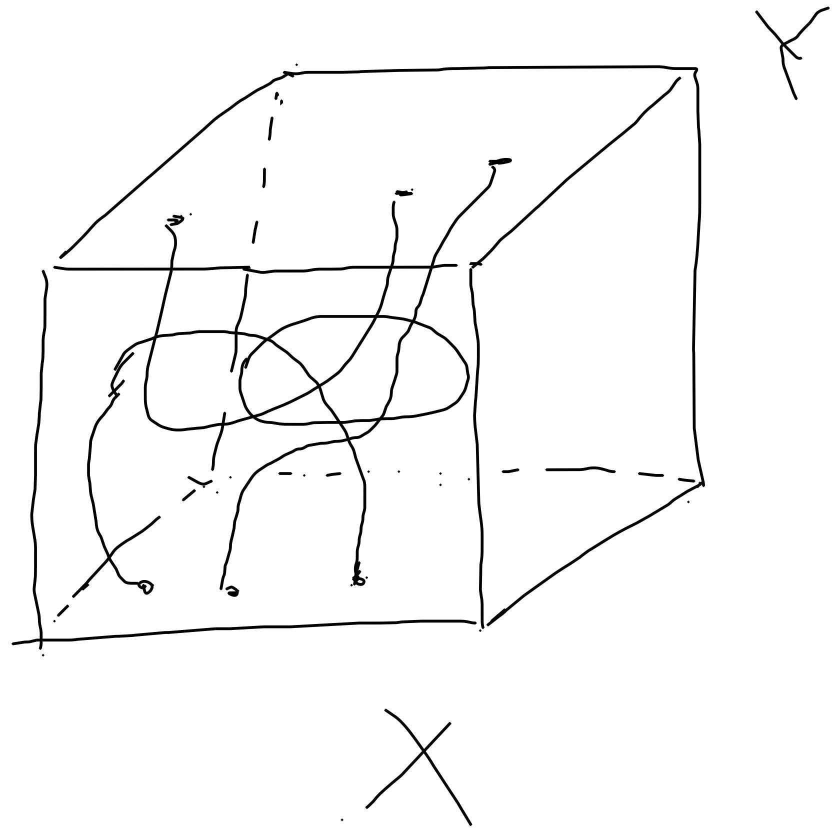
Define a Category $Tang_2$

Objects $(Tang_2)$

a finite number of interior points (particles) in $I^{(2)}$ (including the empty set)



Given two objects X, Y $\hom(X, Y)$ is the Set of all tangles from X to Y.



noninterse~~x~~cting

Smooth arcs embedded in the interior of $I^{(3)}$ joining points of $X \cup Y$ along with circular arcs inside $I^{(3)}$. Also these arcs can only intersect the boundary of $I^{(3)}$ only at the top and at bottom transversally

Composition

Take X, Y as before

Z as