Institute for Advancing Intelligence (IAI), TCG-CREST
Admission Test
Ph.D Program Session: 2021–2022

Date: 29. 05. 2021 Time: $3\frac{1}{2}$ Hours, 9AM to 12:30PM

Note: There are four questions in each of the three groups, total twelve. Each question is of 20 marks. It is expected that an applicant with reasonable preparation should be able to answer around five questions. That is, try to concentrate on five questions initially and then go ahead for further attempts (as much as you can) if you have enough time. Please answer all the parts of a question in the same place. Hurried attempts with "not so logical" arguments should be avoided as those may attract negative credit.

GROUP A: General Mathematics

1. (a) Tabulate the truth tables for all 2-input, 1-output Boolean functions.
   (b) Which of these functions are universal, and which are not? Justify your answer.
   (c) Choose any one of these universal function. Formally prove that any n-input, 1-output Boolean function can be constructed using your chosen function.

   [4+8+8=20]

2. (a) In a football tournament, team Barcelona scores at least one goal in each of the 15 consecutive matches. If they score a total of 23 goals in these 15 matches, prove that there is a sequence of consecutive matches $i, i+1, \ldots, j : 1 \le i \le j \le 15$, where Barcelona scores a total of *exactly* 6 goals.
   (b) Suppose 50 Barcelona fans have been invited to the annual general meeting, and the club decides to choose one of them as the top fan of the year. Given only an unbiased coin (i.e., $\Pr[H] = \Pr[T] = 1/2$), device a strategy to select one of these fans uniformly at random as the top fan. Prove that the probability of selecting any invited fan is exactly $1/50$.

   [10+10=20]

3. Let us consider two n-bit binary strings $x = x_1 \cdots x_n$, and $u = u_1 \cdots u_n$. We define x^u as

$$x^u := \prod_{i=1}^{n} x_i^{u_i},$$

where $a^0 = 1$, for $a \in \{0,1\}$. For a set $X \subseteq \{0,1\}^n$, we define the parity set $\mathcal{P}(X)$ of X as

$$\mathcal{P}(X) := \{u \in \{0,1\}^n : \bigoplus_{x \in X} x^u = 1\},$$

where $\bigoplus$ represents addition modulo 2.

   (a) Show that $x^u = 1$ if and only if $u_i \le x_i$, for all $1 \le i \le n$.
   (b) Let $X \subseteq \{0,1\}^4$. Find $\mathcal{P}(X)$ if (i) $X = \{0011\}$, (ii) $X = \{0,1\}^4$.

(c) For $X \subseteq \{0,1\}^n$, we define the incidence vector of X, denoted as I_X, to be a column vector v of dimension $2^n \times 1$, such that $v[i] = 1$ if and only if the binary representation of i, $\mathsf{bin}(i) \in X$. Let M be a $2^n \times 2^n$ binary matrix such that the entries are indexed by n-bit vectors and defined by

$$M[u, a] = a^u, a, u \in \{0,1\}^n.$$

Prove that for any set $X \subseteq \{0,1\}^n$, $I_{\mathcal{P}(X)} = M \odot I_X$, where $\odot$ represents the matrix multiplication.

[4+8+8=20]

4. (a) In a randomized field experiment, a rectangular block of land is divided into k parallel strips of equal size. There are k varieties of wheat which are allotted to each strip at random. Now consider two specific varieties of wheat, A and B. Find the probability of the event that there are r strips separating those occupied by A and B.

(b) A rare genetic condition affects one in a lakh of the population. A newly developed method detects the condition with an accuracy of 99.9%, that is, the accuracy of the method to detect the condition in a person is 99.9%. Suppose a person tests positive for this condition by this new method. What is the probability that the person actually has the genetic condition? What happens to the probability if the person tests positive for the second consecutive time?

[10+10=20]

GROUP B: Technical Topics in Computer Science

5. (a) An undirected graph is called *k-regular* if all the vertices of the graph have degree k. Draw a 5-*regular* graph with 10 vertices that does not contain any triangle.

(b) There are 10 teams in a round-robin cricket tournament, where every team plays with each of the other teams exactly once. In each round the teams are paired and they play accordingly. Prove that after 4 rounds, there are three teams who have not played against each other.

(c) We call a graph *one-shot draw-able* if it can be drawn in a paper without lifting the pen. Of course, you are not allowed to move your pen through a line that is already drawn. For example, the left-most two graphs shown in Figure. 1 are *one-shot draw-able*, but the rightmost one is not. Given an unordered degree sequence $(d_1, d_2, \ldots, d_n)$, how do you determine whether the graph realizable from this degree sequence is one-shot draw-able or not?

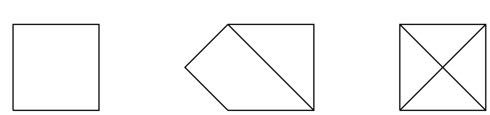

Figure 1: Graphs with degree sequence $(2,2,2,2)$, $(2,2,2,3,3)$, and $(3,3,3,3)$ respectively.

[5+8+7=20]

6. Consider a LIFO (Last-In-First-Out) data structure T, where insertions (I) and deletions (D) are done from the same end. The insertion operation inserts values in T according to a given input sequence, and the deletion operation removes an element from T and outputs the same. Assuming T to be empty at the beginning, an output sequence is a permutation of the input sequence by successive applications of I and D operations. For example, if the input sequence is $(1, 2, 3)$, then the sequence of operations (I,I,D,D,I,D) produce the output sequence $(2, 1, 3)$. However, it is not possible to obtain all the permutations of a given input sequence, e.g, it is impossible to obtain the output sequence $(3, 1, 2)$ corresponding to the input sequence $(1, 2, 3)$.

   (a) Find out the set of impossible output sequences if the input sequence is $(4, 3, 2, 1)$.

   (b) Consider the input sequence $(n, n-1, \ldots, 2, 1)$, and find out the set of all possible output sequences.

   (c) For any input sequence of length n, compute the number of possible output sequences.

[6+6+8=20]

7. Consider that some organization releases data regarding new cases of infection per day, and active cases per day for a period of n days.

   (a) Assume that the values of the new cases per day are distinct, and the maximum new cases in a day is $(n^3 - 1)$. Design a *linear-time* algorithm that sorts these values in decreasing order.

   (b) Suppose the number of active cases per day in an organization follows a special pattern. The number increases strictly up to a certain date and reaches the peak, after that the number decreases strictly. Write an *efficient* algorithm for finding the maximum active cases in a day, during the period of n days. Establish its correctness, and justify the time complexity of the proposed algorithm. [Note: Finding maximum in $O(n)$ time will not fetch any credit.]

[10+10=20]

8. In a small village, there are N houses, numbered $1, \ldots, N$. We say that two houses are *neighbours* if their numbers differ exactly by 1. Suppose you are a house painter and you have been asked to color the houses. The profit for coloring house number i is c_i.

   (a) Suppose you are not allowed to color any neighbouring houses. Devise a polynomial time algorithm that maximizes your profit. For example, assume $N = 5$, and $c_1 = 3, c_2 = 7, c_3 = 9, c_4 = 8, c_5 = 1$, then you should choose to color house number 2 and 4.

   (b) Suppose the profit values are distinct. Write an efficient algorithm to find one profit value which is not among the lowest three profit values.

[14+6=20]

GROUP C: Technical Topics in Mathematics

9. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, periodic function with period 1.

   (a) Show that g is uniformly continuous on $\mathbb{R}$.

   (b) Evaluate $\lim_{m \to 0} \int_0^1 x^{\frac{1}{m}} g(x) dx$.

(c) Show that there exists $s_0 \in \mathbb{R}$ such that

$$g(s_0 - \frac{3\pi}{2}) = g(s_0 + \frac{3\pi}{2}).$$

[5+5+10=20]

10. (a) Consider the vector space $\mathcal{P}_4$, the set of all real-valued polynomials defined on $\mathbb{R}$, with real coefficients and with degree at most 3. Consider the polynomials $p_1, \dots, p_4 \in \mathcal{P}_4$ defined by $p_1(t) = (t^2-1)(t+2)$, $p_2(t) = (t^2-1)(t-2)$, $p_3(t) = (t^2-4)(t+1)$, and $p_4(t) = (t^2-4)(t-1)$. Show that $\{p_1, \dots, p_4\}$ is a basis of $\mathcal{P}_4$.

(b) Consider the following 4×4 real matrix:

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta & \beta & \beta \\ \beta & \alpha & \beta & \beta \\ \beta & \beta & \alpha & \beta \\ \beta & \beta & \beta & \alpha \end{pmatrix}.$$

Find the eigenvalues of $\mathbf{A}$, along with their algebraic multiplicities.

(c) Let V_1 and V_2 be two vector spaces over the field F and $l_1 : V_1 \to V_2$ and $l_2 : V_2 \to V_1$ be linear maps. If $l_1(V_1) = V_2$ and $l_2(V_2) = V_1$, then prove that $\dim(V_1) = \dim(V_2)$.

[6+8+6=20]

11. (a) Let A be an additively written abelian group, and $f,\ g : A \to A$ be two homomorphisms. Define the group homomorphisms $\alpha,\ \beta : A \to A$ by

$$\alpha(a) = a - g(f(a)), \qquad \beta(a) = a - f(g(a)) \quad \text{for all} \ \ a \in A.$$

Prove that the kernel of α is isomorphic to the kernel of β.

(b) Suppose $F \subset K$ is a field extension and $\beta \in K$. If $[F[\beta] : F] = 7$, prove that $F[\beta^2] = F[\beta]$.

[12+8=20]

12. For $n \geq 1$, let

$$S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

with the subspace topology induced from the standard Euclidean topology of $\mathbb{R}^{n+1}$.

(a) Prove that the function $f : \mathbb{R} \to S^1$ defined by $f(t) = e^{2\pi i t}$ is a continuous and open map.

(b) Prove that $S^2 - \{(0,0,1)\}$ is homeomorphic to $\mathbb{R}^2$ by constructing an explicit homeomorphism from $S^2 - \{(0,0,1)\}$ to $\mathbb{R}^2$.

[8+12=20]