Groups and Graphs

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Meet-in-the-Middle

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Groups and Graphs

Different Types of Graphs on Groups

- Cayley Graphs
- Power Graphs

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- Enhanced Power Graph
- (Non)Commuting Graph
- (Non)Generating Graph

Vertices are group elements

Different Types of Graphs on Groups

• Co-Maximal Subgroup Graph

Cayley Graphs
 Power Graphs
 Enhanced Power Graph
 (Non)Commuting Graph
 (Non)Generating Graph
 Intersection Graph of Group
 Vertices are subgroups

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Different Types of Graphs on Groups



P.J. Cameron, *Graphs defined on Groups*, International Journal of Group Theory, 11(2), pp. 53-107, 2022.

Comaximal Subgroup Graph (Akbari et.al.¹)

¹ S. Akbari, B. Miraftab and R. Nikandish, **Co-maximal Graphs of Subgroups of Groups**, **Canadian Math Bulletin**, Vol. 60(1), pp.12-25, 2017.

Definition

Let G be a group and S be the collection of all non-trivial proper subgroups of G. The co-maximal subgroup graph $\Gamma(G)$ of a group G is defined to be a graph with S as the set of vertices and two distinct vertices H and K are adjacent if HK = G. ¹ S. Akbari, B. Miraftab and R. Nikandish, **Co-maximal Graphs of Subgroups of Groups**, **Canadian Math Bulletin**, Vol. 60(1), pp.12-25, 2017.

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Although the definition of comaximal subgroup graph allows the possibility of *G* being infinite, in this talk, we restrict ourselves to finite groups only. The definition implies that the graph is undirected as HK = G if and only if KH = G.

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Examples

Consider the Klein-4 group, K_4 . Then $S = \{H_1 = \{e, a\}, H_2 = \{e, b\}, H_3 = \{e, ab\}\}$ See Figure (A). Next, consider the group S_3 . Then $S = \{H_1 = \{e, (12)\}, H_2 = \{e, (13)\}, H_3 = \{e, (23)\}, H_4 = \{e, (123), (132)\}\}$ See Figure (B).



Examples (Contd.)

Consider $Q_8 = \langle a, b : a^4 = e, a^2 = b^2, ba = a^3b \rangle$. Then $S = \{H_1 = \langle a^2 \rangle, H_2 = \langle a \rangle, H_3 = \langle ab \rangle, H_4 = \langle b \rangle\}$ See Figure (A). Consider $D_4 = \langle a, b : a^4 = e, b^2 = e, ba = a^3b \rangle$. Then $S = \{H_1 = \langle a^2 \rangle, H_2 = \langle b \rangle, H_3 = \langle ab \rangle, H_4 = \langle a^2b \rangle, H_5 = \langle a^3b \rangle, T_1 = \langle a \rangle, T_2 = \{e, a^2, b, a^2b\}, T_3 = \{e, ab, a^2, a^3b\}\}$ See Figure (B).



(A) $\Gamma(Q_8)$ (B) $\Gamma(D_4)$

Figure: Examples of $\Gamma(G)$ (Disconnected Examples)

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A group G is said to be *solvable* if there exists a subnormal series $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_{s-1} \lhd H_s = G$ such that each quotient group H_{i+1}/H_i is abelian and each H_i is normal in H_{i+1} .

Definition

A group G is said to be *supersolvable* if there exists a normal series $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_{s-1} \lhd H_s = G$ such that each quotient group H_{i+1}/H_i is cyclic and each H_i is normal in G.

Definition

A group G is said to be *nilpotent* if there exists a normal series $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_{s-1} \lhd H_s = G$ such that each quotient group $H_{i+1}/H_i \leq Z(G/H_i)$ and each H_i is normal in G.

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Theorem

If G is a finite solvable group, then

- every maximal subgroup of G is of prime-power index in G.
- at least one maximal subgroup is normal in G.

Theorem

If G is a finite nilpotent group, then

- every maximal subgroup of G is normal in G and is of prime index.
- every Sylow subgroup is normal in G.
- G is the direct product of its Sylow subgroups.

Theorem (Akbari, Miraftab, Nikandish)

Let G be a finite group with at least two non-trivial proper subgroups. Then TFAE:

- Γ(G) is connected.
- $\Gamma(G)$ has no isolated vertex.
- $diam(\Gamma(G)) \leq 3$.
- G is supersolvable and its Sylow subgroups are all elementary abelian.

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Theorem (-, Saha, Alkaseasbeh)

Let G be a finite group. If $|\Phi(G)| \neq 1$, then $\Gamma(G)$ has an isolated vertex. If G is nilpotent, then $\Gamma(G)$ has an isolated vertex iff $|\Phi(G)| \neq 1$.

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Key Point

• Maximal subgroups are normal.

The *Frattini subgroup* of a group G, denoted by $\Phi(G)$ is the intersection of all maximal subgroups of G.

Theorem (-, Saha, Alkaseasbeh)

Let G be a finite group. If $|\Phi(G)| \neq 1$, then $\Gamma(G)$ has an isolated vertex. If G is nilpotent, then $\Gamma(G)$ has an isolated vertex iff $|\Phi(G)| \neq 1$.

Remark (Solvability is not enough)

If G is not nilpotent, $|\Phi(G)| = 1$ does not imply $\Gamma(G)$ is isolate-free. For example, A_4 is not nilpotent and $\Phi(A_4)$ is trivial. But $\Gamma(A_4)$ is the disjoint union of a star $K_{1,4}$ and three isolated vertices.

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Remark (Super-Solvability is also not enough)

Let $G = \langle a, b : a^5 = b^4 = 1, ab = ba^2 \rangle$ (Frobenius Group of order 20). *G* is supersolvable and $|\Phi(G)| = 1$, but it has five isolated vertices.

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Theorem (-, Saha, Alkaseasbeh)

Let G be a finite group. If $|\Phi(G)| \neq 1$, then $\Gamma(G)$ has an isolated vertex. If G is nilpotent, then $\Gamma(G)$ has an isolated vertex iff $|\Phi(G)| \neq 1$.

Under what condition, $|\Phi(G)| = 1 \Leftrightarrow$ no isolated vertices? Nilpotent Groups $\subset \bigcirc \subset$ Super-Solvable Groups \subset Solvable Groups

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Let G be a finite group. If G is a cyclic p-group, then $\Gamma(G)$ has no edges. If G is a solvable group, then $\Gamma(G)$ has no edges iff G is a cyclic p-group.

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Image: A matrix

Let G be a finite group. If G is a cyclic p-group, then $\Gamma(G)$ has no edges. If G is a solvable group, then $\Gamma(G)$ has no edges iff G is a cyclic p-group.

Sketch of Proof:

- As G is solvable, by a theorem of Hall, ∃ a Sylow p-subgroup H and Hall p'-subgroup K such that G = HK and H ∩ K = {e}.
- Thus K is trivial and $|G| = p^n$.
- $\exists N \lhd G$ such that $|N| = p^{n-1}$.
- Choose $g \in G \setminus N$ and $A = \langle g \rangle$.
- If A is a proper subgroup of G, we have $A \sim N$. Thus A = G.

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- Choose $g \in G \setminus N$ and $A = \langle g \rangle$.
- If A is a proper subgroup of G, we have $A \sim N$. Thus A = G. Remark The solvability is necessary for the converse part, since if G = PSL(2, 13), then $\Gamma(G)$ consists only of isolated vertices.

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Only Isolated Vertices

Theorem (-, Saha)

Let G be a finite group. Then $\Gamma(G)$ is edgeless if and only if $G/\Phi(G)$ is simple and has no factorization.

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 if H ⊲ G, then H ⊄ M for some maximal subgroup M of G and H ∼ M!

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 if H ⊲ G, then H ⊄ M for some maximal subgroup M of G and H ∼ M!
- Γ(G) is edgeless ⇔ G has no factorization
 ⇔ G ≠ HK for any two maximal subgroups of G
 ⇔ [G : H] > [K : H ∩ K]

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 if H ⊲ G, then H ∉ M for some maximal subgroup M of G and H ∼ M!
- $\Gamma(G)$ is edgeless $\Leftrightarrow G$ has no factorization $\Leftrightarrow G \neq HK$ for any two maximal subgroups of G $\Leftrightarrow [G:H] > [K:H \cap K]$
- As Φ(G) is contained in all maximal subgroups of G, G has no factorization into maximal subgroups
 ⇔ G/Φ(G) has no factorization, i.e., Γ(G/Φ(G)) is edgeless.

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 if H ⊲ G, then H ∉ M for some maximal subgroup M of G and H ∼ M!
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- As Φ(G) is contained in all maximal subgroups of G, G has no factorization into maximal subgroups
 ⇔ G/Φ(G) has no factorization, i.e., Γ(G/Φ(G)) is edgeless.
- As $G/\Phi(G)$ has trivial Fratinni subgroup, $G/\Phi(G)$ is simple.

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Remark: If $\Gamma(G)$ is edgeless, then $G/\Phi(G)$ is simple. Now, two cases may arise:

- If $G/\Phi(G)$ is prime order cyclic group, then G is a cyclic p-group.
- If $G/\Phi(G)$ is a non-abelian simple group with no factorization, then $G/\Phi(G)$ is isomorphic to one of the groups without factorization studied by Liebeck *et.al*.

M.W. Liebeck, C.E. Praeger and J. Saxl, *The maximal factorizations of the finite simple groups and their automorphism groups*, Memoirs Amer. Math. Soc. 86, pp.1-151, 1990.

Let G be a finite group such that G has a maximal subgroup which is normal in G. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the diameter of the component is less than or equal to 4.

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- Let H be a maximal subgroup of G with H ≤ G and L be another maximal subgroup of G.
- $H \sim L$, i.e., H is not an isolated vertex in $\Gamma(G)$.

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- Let H be a maximal subgroup of G with H ≤ G and L be another maximal subgroup of G.
- $H \sim L$, i.e., H is not an isolated vertex in $\Gamma(G)$.
- Let C₁ be the component of Γ(G) which contains H. If possible, let there exists another component C₂ of Γ(G) and H' ~ K' in C₂.
- At least one of H' and K' is not contained in H. Let $H' \subsetneq H$. Then $H \sim H'$ in $\Gamma(G)$



• If *H* is the unique maximal subgroup, then *G* is a cyclic *p*-group, and $\Gamma(G)$ is empty. So, we assume that \exists maximal subgroups of *G*, apart from *H*.

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- Let A, B be two arbitrary vertices of the component.
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- A ⊈ H and B ⊆ H, then as B is not an isolated vertex, B ~ M, for some maximal subgroup M ≠ H. Thus B ~ M ~ H ~ A, i.e., d(A, B) ≤ 3.

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- If $A, B \not\subseteq H$, then we have $A \sim H \sim B$, i.e., $d(A, B) \leq 2$.
- A ⊈ H and B ⊆ H, then as B is not an isolated vertex, B ~ M, for some maximal subgroup M ≠ H. Thus B ~ M ~ H ~ A, i.e., d(A, B) ≤ 3.
- If $A, B \subseteq H$. Clearly $A \not\sim B$. As A, B are not isolated vertices, there exist maximal subgroups M_A, M_B such that $M_A, M_B \neq H$ and $A \sim M_A$ and $B \sim M_B$.
- If $M_A = M_B$, then $A \sim M_A \sim B$. If $M_A \neq M_B$, then $A \sim M_A \sim H \sim M_B \sim B$, i.e., $d(A, B) \leq 4$.

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Let G be a finite group such that G has a maximal subgroup which is normal in G. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the diameter of the component is less than or equal to 4.



Find a group G, for which the component has diameter 4.

Let G be a finite group such that G has a maximal subgroup which is normal in G. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the diameter of the component is less than or equal to 4.

Corollary

Let G be a finite solvable group. Then $\Gamma(G)$ is connected apart from some possible isolated vertices.

Theorem

If G is nilpotent, the diameter of the component is less than or equal to 3.

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Let G be a group. The <u>deleted co-maximal subgroup graph</u> of G, denoted by $\Gamma^*(G)$, is defined as the graph obtained by removing the isolated vertices from $\Gamma(G)$.

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Corollary

Let G be a finite solvable group. Then $\Gamma^*(G)$ is connected. If G is nilpotent, then diameter of $\Gamma^*(G)$ is less than or equal to 3 and $\Gamma^*(G) = \Gamma(G)$ if and only if $\Phi(G)$ is trivial.

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Remark

There exists groups like S_n with $n \ge 5$ which are not solvable but has a maximal subgroup A_n which is normal in S_n . Thus there exists finite non-solvable groups G such that $\Gamma^*(G)$ is connected.

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Remark

There exists simple groups like A_n with $n \ge 5$ which does not have any maximal subgroup which is normal in A_n . But $\Gamma^*(A_5)$ is a connected graph of order 17.

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Question:

Does there exist a finite non-solvable group G such that $\Gamma^*(G)$ is non-empty and disconnected?



Theorem

Let G be a finite group. Then $\Gamma(G)$ has an universal vertex if and only if either G is non-cyclic abelian group of order p^2 or G is group of order pq, where p and q are distinct primes.

Theorem

Let G be a finite nilpotent group. Then $\gamma(\Gamma(G)) = 2$ if and only if G is isomorphic to one of the groups: $\mathbb{Z}_{p^3}, \mathbb{Z}_{p^2q}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p, M_{p^3}, D_4, Q_8, SD_8$.

Theorem

Let G be a finite nilpotent group. Then $\Gamma(G)$ is bipartite if and only if G is a cyclic group of order p^a or p^aq^b , where p, q are distinct primes.

Theorem

Let G be a finite nilpotent group. Then

$$girth(\Gamma^*(G)) = \begin{cases} \infty, & \text{if } G \cong \mathbb{Z}_{p^a} \text{ or } \mathbb{Z}_{p^a q}, \text{ where } a \ge 1\\ 4, & \text{if } G \cong \mathbb{Z}_{p^a q^b}, \text{ where } a, b \ge 2\\ 3, & \text{otherwise.} \end{cases}$$

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Definition

The clique number, $\omega(G)$, of a graph of G is the size of a maximum complete subgraph of G. The chromatic number, $\chi(G)$, of a graph G is the minimum number of colours required to color the vertices such that adjacent vertices receive distinct colours.

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Result

For any graph G, $\chi(G) \ge \omega(G)$.

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Result

For any graph G, $\chi(G) \ge \omega(G)$.

Definition

A graph G is said to be weakly perfect if $\chi(G) = \omega(G)$.

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• It suffices to show that $\chi(\Gamma(G)) \leq \omega(\Gamma(G))$.

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- It suffices to show that χ(Γ(G)) ≤ ω(Γ(G)).
- Let $S = \{H_1, H_2, \dots, H_\omega\}$ be a maximum clique.
- As no two of them are contained in any maximal subgroup, WLOG, we can assume each H_i to be maximal subgroups, i.e., $|Max(G)| \ge \omega$

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- Note that Max(G) may contain maximal subgroups other than those in S. Let M ∈ Max(G) \ S. Thus M ≁ H_i for some i.
- Assign the colour *i* to *M_i* and all of its subgroups which have not been coloured previously. Repeat this process.
- All subgroups receive some colour and it forms a proper colouring, i.e., $\chi(\Gamma(G)) \le \omega(\Gamma(G))$

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Definition

A graph G is said to be perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G.

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• Is $\Gamma(G)$ perfect?

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• When is $\Gamma(G)$ perfect?

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- Γ(Z_n) is perfect iff number of distinct prime factors of n is ≤ 4.
 Γ(D_n) is perfect iff one of the two conditions hold:
 - *n* is odd and $\pi(n) \leq 4$.
 - *n* is even and either $\pi(n) \leq 2$ or $\pi(n) = 3$ and $4 \nmid n$.

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• Characterize G for which $\Gamma(G)$ is perfect?

• Let G be a group such that $\Gamma(G)$ has isolated vertices. A natural question arises whether there always exists a group H such that $\Gamma^*(G) \cong \Gamma(H)$.

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Example

Let $G = \mathbb{Z}_{p^2q^2}$ where p, q are distinct primes. Then $\Gamma(G) \cong C_4 \cup 3K_1$ and $\Gamma^*(G) \cong C_4$. We look for a group H such that $\Gamma(H) \cong C_4$.

Theorem (-,Saha)

There does not exist any finite group H such that $\Gamma(H) \cong C_4$

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Identifying Groups from Graphs

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Note that Γ(ℤ₆) ≅ Γ(ℤ₁₅) ≅ K₂, i.e., non-isomorphic groups can have isomorphic co-maximal subgroup graphs. This leads us to the next question. Under what condition, Γ(G₁) ≅ Γ(G₂) implies G₁ ≅ G₂?

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- If we can not get the group G uniquely from Γ(G), can we get some information about the group G from the graph?
- Are there any class of groups G, for which we can uniquely get the group G from its graph?

Definition

The independence number $\alpha(G)$ of a graph G is the size of a maximum subset of vertices in which no two are adjacent.

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Lemma (-,Saha)

Let G be a finite group such that $\alpha(\Gamma(G)) \leq 8$. Then G is solvable.

Theorem (-,Saha)

Let G be a finite group such that $\Gamma(G) \cong \Gamma(Q_8)$, then $G \cong Q_8$.

Theorem (-,Saha)

Let G be a finite group such that $\Gamma(G) \cong \Gamma(D_{2^k})$, then $G \cong D_{2^k}$.

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Let G be a finite group such that $\Gamma(G) \cong \Gamma(A_4)$, then $G \cong A_4$ or \mathbb{Z}_{p^4q} , where p, q are distinct primes.

Isomorphic Graphs

Definition

Two positive integers *n* and *m* are said to be of same prime-factorization type if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\beta_1} q_2^{\beta_2} \cdots q_k^{\beta_k}$ where p_i, q_i 's are primes and there exists $\sigma \in S_k$ such that $\alpha_i = \beta_{\sigma(i)}$ for i = 1, 2, ..., k.
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Theorem (-,Biswas,Saha)

Let n and m be two integers. Then $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$ if and only if m and n are of same prime-factorization type.

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Let n and m be two integers. If $\Gamma(D_n) \cong \Gamma(D_m)$, then m and n are of same prime-factorization type.

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- Characterize groups G for which |Φ(G)| = 1 implies that Γ(G) has no isolated vertices.
- Characterize G when $\Gamma(G)$ is perfect.
- Which groups G are uniquely determined by $\Gamma(G)$?

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27th April, 2022

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