# Groups and Graphs

**Angsuman Das**

Department of Mathematics,
Presidency University, Kolkata

Meet-in-the-Middle

27th April, 2022

# Different Types of Graphs on Groups

- Cayley Graphs
- Power Graphs
- Enhanced Power Graph
- (Non)Commuting Graph
- (Non)Generating Graph

Vertices are group elements

# Different Types of Graphs on Groups

- Cayley Graphs
- Power Graphs
- Enhanced Power Graph
- (Non)Commuting Graph
- (Non)Generating Graph

Vertices are group elements

- Intersection Graph of Group
- **Co-Maximal Subgroup Graph**

Vertices are subgroups

# Different Types of Graphs on Groups

- Cayley Graphs
- Power Graphs
- Enhanced Power Graph
- (Non)Commuting Graph
- (Non)Generating Graph

Vertices are group elements

- Intersection Graph of Group
- **Co-Maximal Subgroup Graph**

Vertices are subgroups

P.J. Cameron, *Graphs de ned on Groups*, International Journal of Group Theory, 11(2), pp. 53-107, 2022.

# Comaximal Subgroup Graph (Akbari *et.al.*[1])

[1] S. Akbari, B. Miraftab and R. Nikandish, **Co-maximal Graphs of Subgroups of Groups**, **Canadian Math Bulletin**, Vol. 60(1), pp.12-25, 2017.

## Definition

Let $G$ be a group and $S$ be the collection of all non-trivial proper subgroups of $G$. The co-maximal subgroup graph $\Gamma(G)$ of a group $G$ is defined to be a graph with $S$ as the set of vertices and two distinct vertices $H$ and $K$ are adjacent if $HK = G$.

# Comaximal Subgroup Graph (Akbari *et.al.*[1])

[1] S. Akbari, B. Miraftab and R. Nikandish, **Co-maximal Graphs of Subgroups of Groups**, **Canadian Math Bulletin**, Vol. 60(1), pp.12-25, 2017.

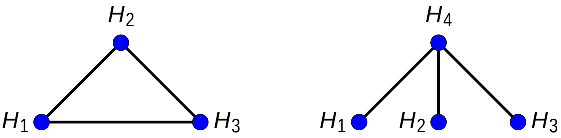
## Definition

Let $G$ be a group and $S$ be the collection of all non-trivial proper subgroups of $G$. The co-maximal subgroup graph $\Gamma(G)$ of a group $G$ is defined to be a graph with $S$ as the set of vertices and two distinct vertices $H$ and $K$ are adjacent if $HK = G$.

Although the definition of comaximal subgroup graph allows the possibility of $G$ being infinite, in this talk, we restrict ourselves to finite groups only. The definition implies that the graph is undirected as $HK = G$ if and only if $KH = G$.

# Examples

Consider the Klein-4 group, $K_4$. Then
$S = fH_1 = fe; ag; H_2 = fe; bg; H_3 = fe; abgg$ See Figure (A).
Next, consider the group $S_3$. Then $S = fH_1 = fe; (12)g; H_2 =$
$fe; (13)g; H_3 = fe; (23)g; H_4 = fe; (123); (132)gg$ See Figure (B).

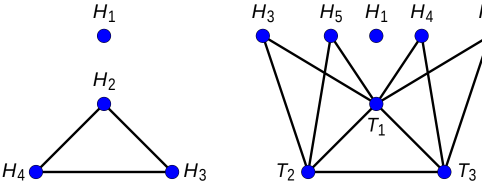



(A) $\Gamma(K_4)$ (B) $\Gamma(S_3)$

Figure: Examples of $\Gamma(G)$ (Connected Examples)

# Examples (Contd.)

Consider $Q_8 = ha; b : a^4 = e; a^2 = b^2; ba = a^3bi$. Then $S = fH_1 = ha^2i; H_2 = hai; H_3 = habi; H_4 = hbig$ See Figure (A).
Consider $D_4 = ha; b : a^4 = e; b^2 = e; ba = a^3bi$. Then $S = fH_1 = ha^2i; H_2 = hbi; H_3 = habi; H_4 = ha^2bi; H_5 = ha^3bi; T_1 = hai; T_2 = fe; a^2; b; a^2bg; T_3 = fe; ab; a^2; a^3bgg$ See Figure (B).




(A) $\Gamma(Q_8)$ (B) $\Gamma(D_4)$

Figure: Examples of $\Gamma(G)$ (Disconnected Examples)

## Definition

A group $G$ is said to be *solvable* if there exists a subnormal series $feg = H_0 \; C \; H_1 \; C \qquad C \; H_{s-1} \; C \; H_s = G$ such that each quotient group $H_{i+1}=H_i$ is abelian and each $H_i$ is normal in $H_{i+1}$.

## Definition

A group $G$ is said to be *supersolvable* if there exists a normal series $feg = H_0 \; C \; H_1 \; C \qquad C \; H_{s-1} \; C \; H_s = G$ such that each quotient group $H_{i+1}=H_i$ is cyclic and each $H_i$ is normal in $G$.

## Definition

A group $G$ is said to be *nilpotent* if there exists a normal series $feg = H_0 \; C \; H_1 \; C \qquad C \; H_{s-1} \; C \; H_s = G$ such that each quotient group $H_{i+1}=H_i \qquad Z(G=H_i)$ and each $H_i$ is normal in $G$.

# For finite groups, Nilpotent ) Supersolvable ) Solvable

**Theorem**

*If $G$ is a nite solvable group, then*

- *every maximal subgroup of $G$ is of prime-power index in $G$.*
- *at least one maximal subgroup is normal in $G$.*

**Theorem**

*If $G$ is a nite nilpotent group, then*

- *every maximal subgroup of $G$ is normal in $G$ and is of prime index.*
- *every Sylow subgroup is normal in $G$.*
- *$G$ is the direct product of its Sylow subgroups.*

# When is $\Gamma(G)$ connected?

**Theorem (Akbari, Miraftab, Nikandish)**

*Let $G$ be a nite group with at least two non-trivial proper subgroups. Then TFAE:*

- $\Gamma(G)$ *is connected.*
- $\Gamma(G)$ *has no isolated vertex.*
- $diam(\Gamma(G))$ 3.
- $G$ *is supersolvable and its Sylow subgroups are all elementary abelian.*

# Isolated vertices of $\Gamma(G)$

## Definition

The *Frattini subgroup* of a group $G$, denoted by $\Phi(G)$ is the intersection of all maximal subgroups of $G$.

# Isolated vertices of $\Gamma(G)$

## Definition

The *Frattini subgroup* of a group $G$, denoted by $\Phi(G)$ is the intersection of all maximal subgroups of $G$.

## Theorem (-, Saha, Alkaseasbeh)

*Let $G$ be a finite group. If $|\Phi(G)| \neq 1$, then $\Gamma(G)$ has an isolated vertex. If $G$ is nilpotent, then $\Gamma(G)$ has an isolated vertex iff $|\Phi(G)| \neq 1$.*

# Isolated vertices of $\Gamma(G)$

**Definition**

The *Frattini subgroup* of a group $G$, denoted by $\Phi(G)$ is the intersection of all maximal subgroups of $G$.

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a finite group. If $|\Phi(G)| \neq 1$, then $\Gamma(G)$ has an isolated vertex. If $G$ is nilpotent, then $\Gamma(G)$ has an isolated vertex iff $|\Phi(G)| \neq 1$.*

Key Point

Maximal subgroups are normal.

# Isolated vertices of $\Gamma(G)$

## Definition

The *Frattini subgroup* of a group $G$, denoted by $\Phi(G)$ is the intersection of all maximal subgroups of $G$.

## Theorem (-, Saha, Alkaseasbeh)

*Let $G$ be a finite group. If $|\Phi(G)| \neq 1$, then $\Gamma(G)$ has an isolated vertex. If $G$ is nilpotent, then $\Gamma(G)$ has an isolated vertex iff $|\Phi(G)| \neq 1$.*

## Remark (Solvability is not enough)

If $G$ is not nilpotent, $|\Phi(G)| = 1$ does not imply $\Gamma(G)$ is isolate-free. For example, $A_4$ is not nilpotent and $\Phi(A_4)$ is trivial. But $\Gamma(A_4)$ is the disjoint union of a star $K_{1,4}$ and three isolated vertices.

# Isolated vertices of $\Gamma(G)$

**Definition**

The *Frattini subgroup* of a group $G$, denoted by $\Phi(G)$ is the intersection of all maximal subgroups of $G$.

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a finite group. If $|\Phi(G)| \neq 1$, then $\Gamma(G)$ has an isolated vertex. If $G$ is nilpotent, then $\Gamma(G)$ has an isolated vertex iff $|\Phi(G)| \neq 1$.*

**Remark (Super-Solvability is also not enough)**

Let $G = \langle a, b : a^5 = b^4 = 1, ab = ba^2 \rangle$ (Frobenius Group of order 20). $G$ is supersolvable and $|\Phi(G)| = 1$, but it has five isolated vertices.

# Isolated vertices of $\Gamma(G)$

**Definition**

The *Frattini subgroup* of a group $G$, denoted by $\Phi(G)$ is the intersection of all maximal subgroups of $G$.

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a finite group. If $|\Phi(G)| \neq 1$, then $\Gamma(G)$ has an isolated vertex. If $G$ is nilpotent, then $\Gamma(G)$ has an isolated vertex iff $|\Phi(G)| \neq 1$.*

**Under what condition, $|\Phi(G)| = 1$ , no isolated vertices?**

Nilpotent Groups Super-Solvable Groups Solvable Groups

# Only Isolated Vertices

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a finite group. If $G$ is a cyclic $p$-group, then $\Gamma(G)$ has no edges. If $G$ is a solvable group, then $\Gamma(G)$ has no edges iff $G$ is a cyclic $p$-group.*

# Only Isolated Vertices

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a finite group. If $G$ is a cyclic $p$-group, then $\Gamma(G)$ has no edges. If $G$ is a solvable group, then $\Gamma(G)$ has no edges iff $G$ is a cyclic $p$-group.*

Sketch of Proof:

As $G$ is solvable, by a theorem of Hall, $\exists$ a Sylow $p$-subgroup $H$ and Hall $p'$-subgroup $K$ such that $G = HK$ and $H \cap K = \{e\}$.

Thus $K$ is trivial and $|G| = p^n$.

$\exists N \lhd G$ such that $|N| = p^{n-1}$.

Choose $g \in G \setminus N$ and $A = \langle g \rangle$.

If $A$ is a proper subgroup of $G$, we have $A \leq N$. Thus $A = G$.

# Only Isolated Vertices

## Theorem (-, Saha, Alkaseasbeh)

*Let $G$ be a finite group. If $G$ is a cyclic $p$-group, then $\Gamma(G)$ has no edges. If $G$ is a solvable group, then $\Gamma(G)$ has no edges iff $G$ is a cyclic $p$-group.*

Sketch of Proof:

- As $G$ is solvable, by a theorem of Hall, $\exists$ a Sylow $p$-subgroup $H$ and Hall $p'$-subgroup $K$ such that $G = HK$ and $H \cap K = \{e\}$.
- Thus $K$ is trivial and $|G| = p^n$.
- $\exists N \lhd G$ such that $|N| = p^{n-1}$.
- Choose $g \in G \setminus N$ and $A = \langle g \rangle$.
- If $A$ is a proper subgroup of $G$, we have $A \subseteq N$. Thus $A = G$.

Remark The solvability is necessary for the converse part, since if $G = PSL(2,13)$, then $\Gamma(G)$ consists only of isolated vertices.

# Only Isolated Vertices

**Theorem (-, Saha)**

*Let $G$ be a nite group. Then $\Gamma(G)$ is edgeless if and only if $G/\Phi(G)$ is simple and has no factorization.*

# Only Isolated Vertices

**Theorem (-, Saha)**

*Let $G$ be a finite group. Then $\Gamma(G)$ is edgeless if and only if $G/\Phi(G)$ is simple and has no factorization.*

Sketch of Proof:

Note: If $\Gamma(G)$ is edgeless and $\Phi(G)$ is trivial, then $G$ is simple.

if $H \lhd G$, then $H \leqslant M$ for some maximal subgroup $M$ of $G$ and $H$ $M$!

# Only Isolated Vertices

**Theorem (-, Saha)**

*Let $G$ be a nite group. Then $\Gamma(G)$ is edgeless if and only if $G/\Phi(G)$ is simple and has no factorization.*

Sketch of Proof:

Note: If $\Gamma(G)$ is edgeless and $\Phi(G)$ is trivial, then $G$ is simple.

if $H \subset G$, then $H \leqslant M$ for some maximal subgroup $M$ of $G$ and $H \quad M$!

$\Gamma(G)$ is edgeless $\Leftrightarrow$ $G$ has no factorization

$\Leftrightarrow$ $G \neq HK$ for any two maximal subgroups of $G$

$\Leftrightarrow$ $[G : H] > [K : H \cap K]$

# Only Isolated Vertices

**Theorem (-, Saha)**

*Let $G$ be a finite group. Then $\Gamma(G)$ is edgeless if and only if $G/\Phi(G)$ is simple and has no factorization.*

Sketch of Proof:

Note: If $\Gamma(G)$ is edgeless and $\Phi(G)$ is trivial, then $G$ is simple.
if $H \trianglelefteq G$, then $H \leq M$ for some maximal subgroup $M$ of $G$ and $H$ $M$!

$\Gamma(G)$ is edgeless $\Leftrightarrow$ $G$ has no factorization
$\Leftrightarrow$ $G \neq HK$ for any two maximal subgroups of $G$
$\Leftrightarrow$ $[G : H] > [K : H \cap K]$

As $\Phi(G)$ is contained in all maximal subgroups of $G$, $G$ has no factorization into maximal subgroups
$\Leftrightarrow$ $G/\Phi(G)$ has no factorization, i.e., $\Gamma(G/\Phi(G))$ is edgeless.

# Only Isolated Vertices

**Theorem (-, Saha)**

*Let $G$ be a finite group. Then $\Gamma(G)$ is edgeless if and only if $G/\Phi(G)$ is simple and has no factorization.*

Sketch of Proof:

Note: If $\Gamma(G)$ is edgeless and $\Phi(G)$ is trivial, then $G$ is simple.

if $H \subset G$, then $H \leq M$ for some maximal subgroup $M$ of $G$ and $H$ $M$!

$\Gamma(G)$ is edgeless $\Leftrightarrow$ $G$ has no factorization

$\Leftrightarrow$ $G \neq HK$ for any two maximal subgroups of $G$

$\Leftrightarrow$ $[G : H] > [K : H \cap K]$

As $\Phi(G)$ is contained in all maximal subgroups of $G$, $G$ has no factorization into maximal subgroups

$\Leftrightarrow$ $G/\Phi(G)$ has no factorization, i.e., $\Gamma(G/\Phi(G))$ is edgeless.

As $G/\Phi(G)$ has trivial Fratinni subgroup, $G/\Phi(G)$ is simple.

# Only Isolated Vertices

Remark: If $\Gamma(G)$ is edgeless, then $G/\Phi(G)$ is simple. Now, two cases may arise:

- If $G/\Phi(G)$ is prime order cyclic group, then $G$ is a cyclic $p$-group.
- If $G/\Phi(G)$ is a non-abelian simple group with no factorization, then $G/\Phi(G)$ is isomorphic to one of the groups without factorization studied by Liebeck *et.al.*

M.W. Liebeck, C.E. Praeger and J. Saxl, *The maximal factorizations of the finite simple groups and their automorphism groups*, Memoirs Amer. Math. Soc. 86, pp.1-151, 1990.

# Ignoring the isolated vertices

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a nite group such that $G$ has a maximal subgroup which is normal in $G$. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the diameter of the component is less than or equal to 4.*

# Ignoring the isolated vertices

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a nite group such that $G$ has a maximal subgroup which is normal in $G$. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the diameter of the component is less than or equal to $4$.*

Let $H$ be a maximal subgroup of $G$ with $H$ E $G$ and $L$ be another maximal subgroup of $G$.

$H$ $L$, i.e., $H$ is not an isolated vertex in $\Gamma(G)$.

# Ignoring the isolated vertices

## Theorem (-, Saha, Alkaseasbeh)

*Let $G$ be a nite group such that $G$ has a maximal subgroup which is normal in $G$. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the diameter of the component is less than or equal to 4.*

Let $H$ be a maximal subgroup of $G$ with $H$ E $G$ and $L$ be another maximal subgroup of $G$.

$H$ $L$, i.e., $H$ is not an isolated vertex in $\Gamma(G)$.

Let $C_1$ be the component of $\Gamma(G)$ which contains $H$. If possible, let there exists another component $C_2$ of $\Gamma(G)$ and $H^0$ $K^0$ in $C_2$.

At least one of $H^0$ and $K^0$ is not contained in $H$. Let $H^0$ ( $H$. Then $H$ $H^0$ in $\Gamma(G)$

# Ignoring the isolated vertices

**Proof Continued:**

If $H$ is the unique maximal subgroup, then $G$ is a cyclic $p$-group, and $(G)$ is empty. So, we assume that ∃ maximal subgroups of $G$, apart from $H$.

# Ignoring the isolated vertices

Proof Continued:

If $H$ is the unique maximal subgroup, then $G$ is a cyclic $p$-group, and $(G)$ is empty. So, we assume that $\exists$ maximal subgroups of $G$, apart from $H$.

Let $A, B$ be two arbitrary vertices of the component.

**Proof Continued:**

If $H$ is the unique maximal subgroup, then $G$ is a cyclic $p$-group, and $\Gamma(G)$ is empty. So, we assume that $\exists$ maximal subgroups of $G$, apart from $H$.

Let $A, B$ be two arbitrary vertices of the component.

If $A, B \not\leq H$, then we have $A \sim H \sim B$, i.e., $d(A, B) \leq 2$.

Proof Continued:

- If $H$ is the unique maximal subgroup, then $G$ is a cyclic $p$-group, and $(G)$ is empty. So, we assume that 9 maximal subgroups of $G$, apart from $H$.
- Let $A; B$ be two arbitrary vertices of the component.
- If $A; B$ 6 $H$, then we have $A \quad H \quad B$, i.e., $d(A; B) \quad 2$.
- $A$ 6 $H$ and $B \quad H$, then as $B$ is not an isolated vertex, $B \quad M$, for some maximal subgroup $M$ 6 $H$. Thus $B \quad M \quad H \quad A$, i.e., $d(A; B) \quad 3$.

Proof Continued:

If $H$ is the unique maximal subgroup, then $G$ is a cyclic $p$-group, and $\Gamma(G)$ is empty. So, we assume that $9$ maximal subgroups of $G$, apart from $H$.

Let $A; B$ be two arbitrary vertices of the component.

If $A; B$ 6 $H$, then we have $A \quad H \quad B$, i.e., $d(A; B) \quad 2$.

$A$ 6 $H$ and $B \quad H$, then as $B$ is not an isolated vertex, $B \quad M$, for some maximal subgroup $M \neq H$. Thus $B \quad M \quad H \quad A$, i.e., $d(A; B) \quad 3$.

If $A; B \quad H$. Clearly $A$ 6 $B$. As $A; B$ are not isolated vertices, there exist maximal subgroups $M_A; M_B$ such that $M_A; M_B \neq H$ and $A \quad M_A$ and $B \quad M_B$.

If $M_A = M_B$, then $A \quad M_A \quad B$. If $M_A \neq M_B$, then $A \quad M_A \quad H \quad M_B \quad B$, i.e., $d(A; B) \quad 4$.

# Ignoring the isolated vertices

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a nite group such that $G$ has a maximal subgroup which is normal in $G$. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the diameter of the component is less than or equal to 4.*

Find a group $G$, for which the component has diameter 4.

# Ignoring the isolated vertices

**Theorem (-, Saha, Alkaseasbeh)**

*Let $G$ be a nite group such that $G$ has a maximal subgroup which is normal in $G$. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the diameter of the component is less than or equal to 4.*

**Corollary**

*Let $G$ be a nite solvable group. Then $\Gamma(G)$ is connected apart from some possible isolated vertices.*

**Theorem**

*If $G$ is nilpotent, the diameter of the component is less than or equal to 3.*

# Deleted Co-Maximal Subgroup Graph

## Definition

Let $G$ be a group. The *deleted co-maximal subgroup graph* of $G$, denoted by $\Gamma\ (G)$, is defined as the graph obtained by removing the isolated vertices from $\Gamma(G)$.

# Deleted Co-Maximal Subgroup Graph

## Definition

Let $G$ be a group. The *deleted co-maximal subgroup graph* of $G$, denoted by $\Gamma\ (G)$, is defined as the graph obtained by removing the isolated vertices from $\Gamma(G)$.

## Corollary

*Let $G$ be a  nite solvable group. Then $\Gamma\ (G)$ is connected. If $G$ is nilpotent, then diameter of $\Gamma\ (G)$ is less than or equal to 3 and $\Gamma\ (G) = \Gamma(G)$ if and only if $\Phi(G)$ is trivial.*

# Deleted Co-Maximal Subgroup Graph

## Definition

Let $G$ be a group. The *deleted co-maximal subgroup graph* of $G$, denoted by $\Gamma\ (G)$, is defined as the graph obtained by removing the isolated vertices from $\Gamma(G)$.

## Corollary

*Let $G$ be a nite solvable group. Then $\Gamma\ (G)$ is connected. If $G$ is nilpotent, then diameter of $\Gamma\ (G)$ is less than or equal to $3$ and $\Gamma\ (G) = \Gamma(G)$ if and only if $\Phi(G)$ is trivial.*

## Remark

There exists groups like $S_n$ with $n \quad 5$ which are not solvable but has a maximal subgroup $A_n$ which is normal in $S_n$. Thus there exists finite non-solvable groups $G$ such that $\Gamma\ (G)$ is connected.

# Deleted Co-Maximal Subgroup Graph

## Definition

Let $G$ be a group. The *deleted co-maximal subgroup graph* of $G$, denoted by $\Gamma\ (G)$, is defined as the graph obtained by removing the isolated vertices from $\Gamma(G)$.

## Corollary

*Let $G$ be a nite solvable group. Then $\Gamma\ (G)$ is connected. If $G$ is nilpotent, then diameter of $\Gamma\ (G)$ is less than or equal to 3 and $\Gamma\ (G) = \Gamma(G)$ if and only if $\Phi(G)$ is trivial.*

## Remark

There exists simple groups like $A_n$ with $n$ 5 which does not have any maximal subgroup which is normal in $A_n$. But $\Gamma\ (A_5)$ is a connected graph of order 17.

# Deleted Co-Maximal Subgroup Graph

## Definition

Let $G$ be a group. The *deleted co-maximal subgroup graph* of $G$, denoted by $\Gamma\ (G)$, is defined as the graph obtained by removing the isolated vertices from $\Gamma(G)$.

## Corollary

*Let $G$ be a nite solvable group. Then $\Gamma\ (G)$ is connected. If $G$ is nilpotent, then diameter of $\Gamma\ (G)$ is less than or equal to 3 and $\Gamma\ (G) = \Gamma(G)$ if and only if $\Phi(G)$ is trivial.*

**Question:**

Does there exist a finite non-solvable group $G$ such that $\Gamma\ (G)$ is non-empty and disconnected?

**Theorem**

*Let $G$ be a nite group. Then $\Gamma(G)$ has an universal vertex if and only if either $G$ is non-cyclic abelian group of order $p^2$ or $G$ is group of order $pq$, where $p$ and $q$ are distinct primes.*

**Theorem**

*Let $G$ be a nite nilpotent group. Then $(\Gamma(G)) = 2$ if and only if $G$ is isomorphic to one of the groups: $Z_{p^3}; Z_{p^2q}; Z_{p^2} \quad Z_p; M_{p^3}; D_4; Q_8; SD_8$.*

**Theorem**

*Let $G$ be a finite nilpotent group. Then $\Gamma(G)$ is bipartite if and only if $G$ is a cyclic group of order $p^a$ or $p^a q^b$, where $p, q$ are distinct primes.*

**Theorem**

*Let $G$ be a finite nilpotent group. Then*

$$girth(\Gamma(G)) = \begin{cases} \infty; & \text{if } G = \mathbb{Z}_{p^a} \text{ or } \mathbb{Z}_{p^a q}; \text{ where } a \geq 1 \\ 4; & \text{if } G = \mathbb{Z}_{p^a q^b}; \text{ where } a, b \geq 2 \\ 3; & \text{otherwise.} \end{cases}$$

# Colouring (G)

[redacted]

The clique number, (G), of a graph G is the size of a maximum complete subgraph of G.

The chromatic number, (G), of a graph G is the minimum number of colours required to color the vertices such that adjacent vertices receive distinct colours.

# Colouring (G)

[redacted]

The clique number! (G), of a graph of $G$ is the size of a maximum complete subgraph of $G$.
The chromatic number, (G), of a graph $G$ is the minimum number of colours required to color the vertices such that adjacent vertices receive distinct colours.

[redacted]

For any graph $G$, (G) ! (G).

# Colouring (G)

[redacted]

The clique number! (G), of a graph of G is the size of a maximum complete subgraph of G.
The chromatic number, (G), of a graph G is the minimum number of colours required to color the vertices such that adjacent vertices receive distinct colours.

[redacted]

For any graph G, (G) ! (G).

[redacted]

A graph G is said to be weakly perfect if (G) = ! (G).

# For any nite group $G$, $( G)$ is weakly perfect.

# For any nite groupG, ( G) is weakly perfect.

It su ces to show that ( ( G)) ! ( ( G)).

# For any finite group $G$, $\Gamma(G)$ is weakly perfect.

- It suffices to show that $\chi(\Gamma(G)) \leq \omega(\Gamma(G))$.
- Let $S = \{H_1, H_2, \ldots, H_\omega\}$ be a maximum clique.
- As no two of them are contained in any maximal subgroup, WLOG, we ca assume each $H_i$ to be maximal subgroups, i.e., $|Max(G)| \geq \omega$

# For any finite group $G$, $\Gamma(G)$ is weakly perfect.

- It suffices to show that $\chi(\Gamma(G)) \leq \omega(\Gamma(G))$.
- Let $S = \{H_1, H_2, \ldots, H_\omega\}$ be a maximum clique.
- As no two of them are contained in any maximal subgroup, WLOG, we ca assume each $H_i$ to be maximal subgroups, i.e., $|Max(G)| \geq \omega$.
- Assign the colour 1 to $H_1$ and all proper subgroups of $H_1$.

# For any finite group $G$, $\Gamma(G)$ is weakly perfect.

It suffices to show that $(\Gamma(G)) \quad \omega(\Gamma(G))$.

Let $S = \{H_1, H_2, \ldots, H_\omega\}$ be a maximum clique.

As no two of them are contained in any maximal subgroup, WLOG, we can assume each $H_i$ to be maximal subgroups, i.e., $|Max(G)| \quad \omega$

Assign the colour 1 to $H_1$ and all proper subgroups of $H_1$.

Assign the colour 2 to $H_2$ and all proper subgroups of $H_2$ which are not contained in $H_1$.

# For any finite group $G$, $\Gamma(G)$ is weakly perfect.

It suffices to show that $(\Gamma(G)) \quad !(\Gamma(G))$.

Let $S = fH_1; H_2; :::; H_! g$ be a maximum clique.

As no two of them are contained in any maximal subgroup, WLOG, we can assume each $H_i$ to be maximal subgroups, i.e., $jMax(G)j \quad !$

Assign the colour 1 to $H_1$ and all proper subgroups of $H_1$.

Assign the colour 2 to $H_2$ and all proper subgroups of $H_2$ which are not contained in $H_1$.

For each $i = 1; 2; :::; !$, we assign the colour $i$ to $H_i$ and all proper subgroups of $H_i$ which are not contained in $H_j$, for $j < i$.

# For any finite group $G$, $\Gamma(G)$ is weakly perfect.

It suffices to show that $\chi(\Gamma(G)) \leq \omega(\Gamma(G))$.

Let $S = \{H_1, H_2, \ldots, H_\omega\}$ be a maximum clique.

As no two of them are contained in any maximal subgroup, WLOG, we can assume each $H_i$ to be maximal subgroups, i.e., $|Max(G)| \geq \omega$

Assign the colour 1 to $H_1$ and all proper subgroups of $H_1$.

Assign the colour 2 to $H_2$ and all proper subgroups of $H_2$ which are not contained in $H_1$.

For each $i = 1, 2, \ldots, \omega$, we assign the colour $i$ to $H_i$ and all proper subgroups of $H_i$ which are not contained in $H_j$, for $j < i$.

Note that $Max(G)$ may contain maximal subgroups other than those in $S$. Let $M \in Max(G) \setminus S$. Thus $M \not\subset H_i$ for some $i$.

# For any finite group $G$, $\Gamma(G)$ is weakly perfect.

It suffices to show that $\chi(\Gamma(G)) \leq \omega(\Gamma(G))$.

Let $S = \{H_1, H_2, \ldots, H_\omega\}$ be a maximum clique.

As no two of them are contained in any maximal subgroup, WLOG, we can assume each $H_i$ to be maximal subgroups, i.e., $|Max(G)| \geq \omega$

Assign the colour 1 to $H_1$ and all proper subgroups of $H_1$.

Assign the colour 2 to $H_2$ and all proper subgroups of $H_2$ which are not contained in $H_1$.

For each $i = 1, 2, \ldots, \omega$, we assign the colour $i$ to $H_i$ and all proper subgroups of $H_i$ which are not contained in $H_j$, for $j < i$.

Note that $Max(G)$ may contain maximal subgroups other than those in $S$. Let $M \in Max(G) \setminus S$. Thus $M \not\subset H_i$ for some $i$.

Assign the colour $i$ to $M_i$ and all of its subgroups which have not been coloured previously. Repeat this process.

All subgroups receive some colour and it forms a proper colouring, i.e., $\chi(\Gamma(G)) \leq \omega(\Gamma(G))$

[redacted]

A graph G is said to be perfect if (H) = ! (H) for all induced subgraphs H of G.

# Colouring (G)

[redacted]

A graph G is said to be perfect if (H) = ! (H) for all induced subgraphs H of G.

Is (G) perfect?

# Colouring (G)

[redacted]

A graph $G$ is said to be perfect if $(H) = \, ! \, (H)$ for all induced subgraphs $H$ of $G$.

Is $(G)$ perfect? NO. Consider $Z_n$, $n$ has at least 5 distinct prime factors.

**Definition**

A graph $G$ is said to be perfect if $(H) = !(H)$ for all induced subgraphs $H$ of $G$.

Is $\Gamma(G)$ perfect? NO. Consider $\mathbb{Z}_n$, $n$ has at least 5 distinct prime factors.

When is $\Gamma(G)$ perfect?

$\Gamma(\mathrm{Z}_n)$ is perfect iff number of distinct prime factors of $n$ is 4.

$\Gamma(D_n)$ is perfect iff one of the two conditions hold:

- $n$ is odd and $(n)$ 4.
- $n$ is even and either $(n)$ 2 or $(n) = 3$ and 4 - $n$.

# When $\Gamma(G)$ is perfect

$\Gamma(\mathbb{Z}_n)$ is perfect iff number of distinct prime factors of $n$ is 4.

$\Gamma(D_n)$ is perfect iff one of the two conditions hold:

- $n$ is odd and $(n)$ 4.
- $n$ is even and either $(n)$ 2 or $(n) = 3$ and 4 - $n$.

Characterize $G$ for which $\Gamma(G)$ is perfect?

# Why Isolated Vertices are Important?

Let $G$ be a group such that $\Gamma(G)$ has isolated vertices. A natural question arises whether there always exists a group $H$ such that $\Gamma\ (G) = \Gamma(H)$.

# Why Isolated Vertices are Important?

Let $G$ be a group such that $(G)$ has isolated vertices. A natural question arises whether there always exists a group $H$ such that $(G) = (H)$. The answer is NO.

Let G be a group such that (G) has isolated vertices. A natural question arises whether there always exists a group H such that (G) = ( H). The answer is NO.

[redacted]

Let G = $Z_{p^2q^2}$ where $p; q$ are distinct primes. Then (G) = $C_4$ [ $3K_1$ and (G) = $C_4$. We look for a group H such that (H) = $C_4$.

[redacted]

There does not exist any nite group H such that ( H) = $C_4$

# Identifying Groups from Graphs

# Identifying Groups from Graphs

Note that $(Z_6) = (Z_{15}) = K_2$, i.e., non-isomorphic groups can have isomorphic co-maximal subgroup graphs. This leads us to the next question. Under what condition, $(G_1) = (G_2)$ implies $G_1 = G_2$?

# Identifying Groups from Graphs

Note that $(Z_6) = (Z_{15}) = K_2$, i.e., non-isomorphic groups can have isomorphic co-maximal subgroup graphs. This leads us to the next question. Under what condition, $(G_1) = (G_2)$ implies $G_1 = G_2$?

If we can not get the group $G$ uniquely from $(G)$, can we get some information about the group $G$ from the graph?

# Identifying Groups from Graphs

- Note that $(Z_6) = (Z_{15}) = K_2$, i.e., non-isomorphic groups can have isomorphic co-maximal subgroup graphs. This leads us to the next question. Under what condition, $(G_1) = (G_2)$ implies $G_1 = G_2$?
- If we can not get the group $G$ uniquely from $(G)$, can we get some information about the group $G$ from the graph?
- Are there any class of groups $G$, for which we can uniquely get the group $G$ from its graph?

# Identifying Groups from Graphs

[redacted]

The independence number(G) of a graph $G$ is the size of a maximum subset of vertices in which no two are adjacent.

**Definition**

The independence number $(G)$ of a graph $G$ is the size of a maximum subset of vertices in which no two are adjacent.

**Lemma (-,Saha)**

*Let $G$ be a nite group such that $(\Gamma(G))$ $8$. Then $G$ is solvable.*

# Identifying Groups from Graphs

**Theorem (-,Saha)**

*Let G be a  nite group such that* $\Gamma(G) = \Gamma(Q_8)$, *then* $G = Q_8$.

**Theorem (-,Saha)**

*Let G be a  nite group such that* $\Gamma(G) = \Gamma(D_{2^k})$, *then* $G = D_{2^k}$.

# Identifying Groups from Graphs

**Theorem (-,Saha)**

*Let $G$ be a  nite group such that $\Gamma(G) = \Gamma(Q_8)$, then $G = Q_8$.*

**Theorem (-,Saha)**

*Let $G$ be a  nite group such that $\Gamma(G) = \Gamma(D_{2^k})$, then $G = D_{2^k}$.*

**Theorem (-,Saha)**

*Let $G$ be a  nite group such that $\Gamma(G) = \Gamma(A_4)$, then $G = A_4$ or $Z_{p^4 q}$, where $p; q$ are distinct primes.*

## Definition

Two positive integers $n$ and $m$ are said to be of same prime-factorization type if $n = p_1 \; {}^{1} p_2 \; {}^{2} \quad p_k \; {}^{k}$ and $m = q_1 \; {}^{1} q_2 \; {}^{2} \quad q_k \; {}^{k}$ where $p_i; q_i$'s are primes and there exists $\; 2 \; S_k$ such that $\; {}_{i} = \; {}_{(i)}$ for $i = 1; 2; : : : ; k$.

# Isomorphic Graphs

### Definition

Two positive integers $n$ and $m$ are said to be of same prime-factorization type if $n = p_1 \;^{1} p_2 \;^{2} \quad p_k \;^{k}$ and $m = q_1 \;^{1} q_2 \;^{2} \quad q_k \;^{k}$ where $p_i; q_i$'s are primes and there exists $\;2\; S_k$ such that $\;_i = \;_{(i)}$ for $i = 1; 2; :::; k$.

### Theorem (-,Biswas,Saha)

*Let $n$ and $m$ be two integers. Then $\Gamma(Z_n) = \Gamma(Z_m)$ if and only if $m$ and $n$ are of same prime-factorization type.*

# Isomorphic Graphs

[redacted]

Two positive integers $n$ and $m$ are said to be of same prime-factorization type if $n = p_1^{\ 1} p_2^{\ 2} \quad p_k^{\ k}$ and $m = q_1^{\ 1} q_2^{\ 2} \quad q_k^{\ k}$ where $p_i ; q_i$'s are primes and there exists 2 $S_k$ such that $_i = \quad _{(i)}$ for $i = 1; 2; :::; k$.

[redacted]

Let $n$ and $m$ be two integers. Then$( Z_n) = \ ( Z_m)$ if and only if $m$ and $n$ are of same prime-factorization type.

[redacted]

Let $n$ and $m$ be two integers. If $( D_n) = \ ( D_m)$, then $m$ and $n$ are of same prime-factorization type.

[redacted]

Two positive integers $n$ and $m$ are said to be of same prime-factorization type if $n = p_1{}^{1}p_2{}^{2}\ \ p_k{}^{k}$ and $m = q_1{}^{1}q_2{}^{2}\ \ q_k{}^{k}$ where $p_i; q_i$'s are primes and there exists 2 $S_k$ such that $_i = \ _{(i)}$ for $i = 1; 2; :::; k$.

[redacted]

Let $n$ and $m$ be two integers. Then( $Z_n$) = ( $Z_m$) if and only if $m$ and $n$ are of same prime-factorization type.

[redacted]

Let $n$ and $m$ be two integers. If ( $D_n$) = ( $D_m$), then $m$ and $n$ are of same prime-factorization type.

Does there exists a group $G$, such that $\text{diam}(\quad(G)) = 4$?

# Summarizing the questions

- Does there exists a group $G$, such that diam( (G)) = 4?
- Does there exist a nite non-solvable group $G$ such that (G) is non-empty and disconnected?

# Summarizing the questions

- Does there exists a group $G$, such that $\mathrm{diam}( \; (G)) = 4$?
- Does there exist a finite non-solvable group $G$ such that $\; (G)$ is non-empty and disconnected?
- Characterize groups $G$ for which $|\;(G)| = 1$ implies that $\;(G)$ has no isolated vertices.

- Does there exists a group $G$, such that $\mathrm{diam}(\ (G)) = 4$?
- Does there exist a nite non-solvable group $G$ such that $(G)$ is non-empty and disconnected?
- Characterize groups $G$ for which $|\ (G)| = 1$ implies that $(G)$ has no isolated vertices.
- Characterize $G$ when $(G)$ is perfect.

# Summarizing the questions

- Does there exists a group $G$, such that $\mathrm{diam}(\ (G)) = 4$?
- Does there exist a nite non-solvable group $G$ such that $(G)$ is non-empty and disconnected?
- Characterize groups $G$ for which $|\ (G)| = 1$ implies that $(G)$ has no isolated vertices.
- Characterize $G$ when $(G)$ is perfect.
- Which groups $G$ are uniquely determined by $(G)$?

# References


S. Akbari, B. Miraftab and R. Nikandish, Co-maximal Graphs of Subgroups of Groups, Canadian Math Bulletin, Vol. 60(1), pp.12-25, 2017.

A. Das, M. Saha and S. Al-Kaseasbeh, On Comaximal Subgroup Graph of a Group, Arxiv.

A. Das and M. Saha, On Comaximal Subgroup Graph of a Group - II, Work in Progress.

M. Saha, S. Biswas, A. Das, On Comaximal Subgroup Graph of $Z_n$, International Journal of Group Theory, *In press*.