① SAT ② 3SAT ③ Complete Subgraph

④ Independent Set problem is NP-complete.

Given a graph G & an integer k, decide if G has an independent set of size k.

$IS = \{ \langle G \rangle k$: G has an independent set of size $k \}$

① IS is in NP.

Trivial

② To show that IS is NP-hard we shall reduce $CS \leq_p IS$

Lemma Let $G = (V, E)$ be a graph.

Then $S \subseteq V$ forms a complete subgraph in G iff $V - S$ is an independent set in G.

Pf Let S forms a complete subgraph in G.

Claim $V - S$ is independent in G.

$u, v \notin S$

Suppose.

Vertex Cover

Given a graph G & an integer k, decide if G has a vertex cover of size k.

1. VC $\in$ NP.
2. VC is NP-hard, we will show

$$CS \le_p VC$$

Lemma Let $G = (V, E)$ be a graph & $\overline{G} = (V, \overline{E})$ be its complement

Then $S \subseteq V$ forms a complete subgraph of G iff $V - S$ is a vertex cover of $\overline{G}$

Pf. S forms a complete subgraph of G.

Claim $V - S$ is a vertex cover of $\overline{G}$

Let $\{u, v\} \in \overline{E}$

$\Rightarrow$ $\{u, v\}$ is not in E

$\Rightarrow$ Both u & V cannot be in S

$\Rightarrow$ one of $u, v \notin S$

Conversely suppose $V - S$ is a vertex cover of $\overline{G}$

Let ~~$\{$~~ $u, v \in S$ If $\{u, v\} \notin E$

$\Rightarrow \{u, v\} \in \overline{G}$

Since $V-S$ is a vertex cover of $\overline{G}$
one of u,v is in $V-S$ ✗

$CS \leq_p VC$

$$(G,k) \longmapsto (\overline{G}, n-k), \quad |V| = n.$$

5. Set Cover.

Given a family of sets $\Delta = \{A_1, \ldots A_n\}$ & an integer k, decide if $\exists$ a subfamily $\Gamma = \{A_{i_1}, \ldots, A_{i_k}\}$ s.t. $A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}$

$$= \bigcup_{i=1}^{n} A_i$$

Thm Set Cover is NP-complete.

Pf $SC \in NP$

We shall show that

$$VC \leq_p SC.$$

Fix a graph $G = (V, E)$ & an integer k.

$1 \leq i \leq n$, let $V = \{v_1, v_2, \ldots, v_n\}$.

Consider the sets A_i for $1 \le i \le n$

$$A_i = \left\{ (v_i, v_j), (v_j, v_i) \;\middle|\; \{v_i, v_j\} \in E \text{ for some } j \right\}.$$

$$\Delta = \{ A_i : 1 \le i \le n \}$$

<u>Claim</u> $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ is a vertex cover of G iff $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$ is a set cover of Δ.

Thm Independent Set problem is NP-complete.

Pf ① $IS \in NP$ Trivial.

② We shall prove that $VC \le_p IS$.

Lemma Let $G = (V, E)$ be a graph & let $S \subseteq V$.

Then S is a vertex cover of G iff $V - S$ is an independent set in G.

Pf. Let S be a vertex cover.

$u, v \in V - S$

$\{u, v\} \notin E$

Conversely, if $u, v \in V - S$

$\{u, v\}$ is not an edge of G.

Def^n. Let $G = (V, E)$ be a graph & k be an integer.

By a k-colouring of G we mean a colouring of the vertices of G by one of the k-colours s.t. if $\{u, v\} \in E$ then u & v must receive different colours.

Graph - Coloring Problem

Given a graph $G = (V, E)$ & an integer k, decide if G is k-colorable.

Thm Graph Coloring problem is NP-complete.

Pf. 1. GC is in NP

To show GC is NP-hard we will prove that

$$3SAT \leq_p GC.$$

Our aim is to show, given a formula φ in 3CNF, there is a poly-time transformation that converts φ into a graph (G_φ, k) s.t.

φ is satisfiable iff G_φ is k-colorable

$$\varphi \xrightarrow{T} G_\varphi, k.$$

$$\varphi \in 3SAT \iff (G_\varphi, k) \in GC$$

Let $\varphi = C_1 \wedge C_2 \wedge \dots \wedge C_k$.

Let $x_1, x_2, \dots, x_n$ be the atoms in φ.

Let the vertices of G_φ be

(1) $\{v_i : 1 \leq i \leq n\}$

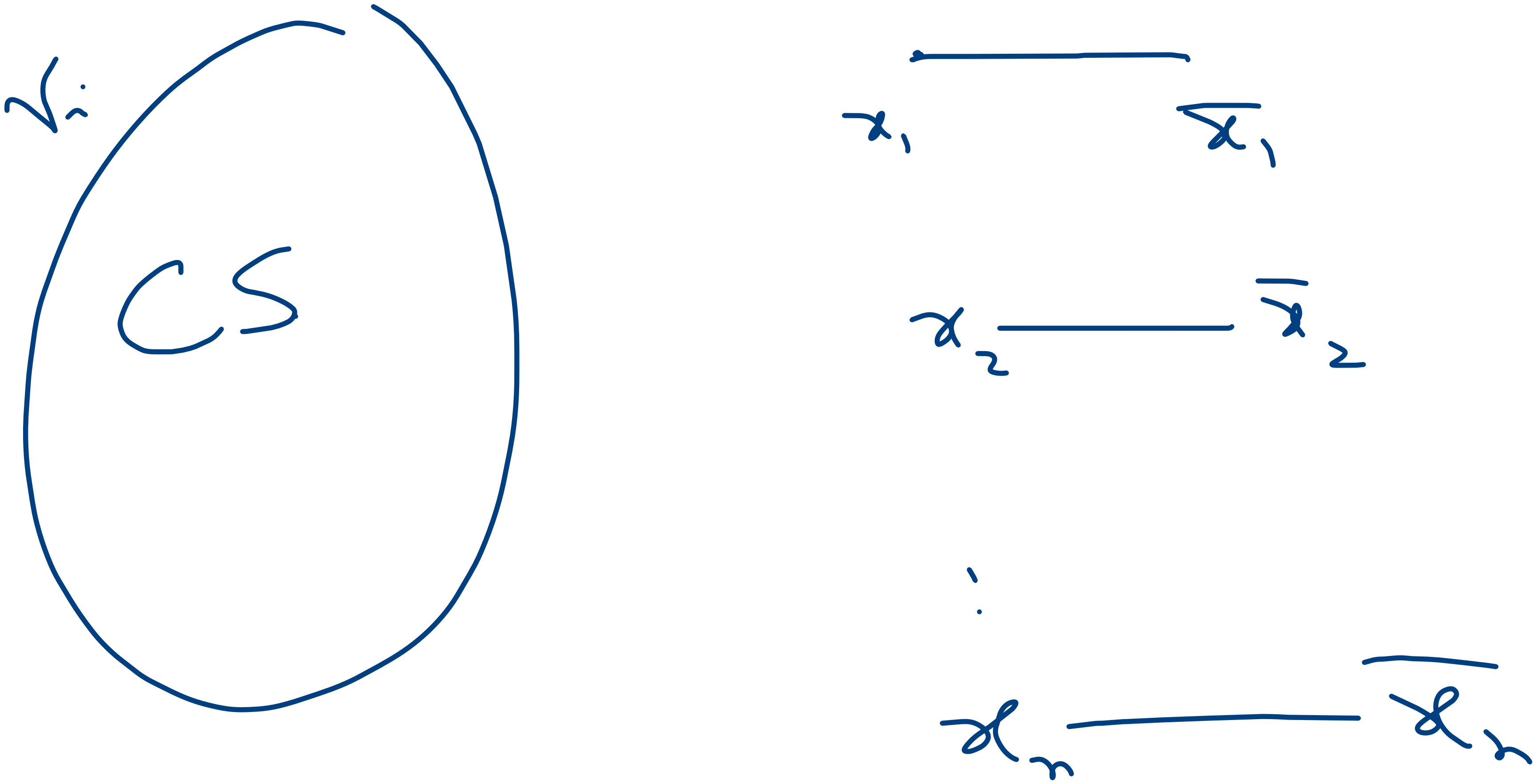
(2) $\{x_i, \overline{x_i} : 1 \leq i \leq n\}$

(3) $\{C_i : \alpha \text{ is a literal}, 1 \leq i \leq n\}$.

The edges of G_φ are as follows:

1. $\{v_i, v_j\}\ 1 \leq i \neq j \leq n$

2. $\{x_i, \overline{x_i}\} : 1 \leq i \leq n$.



(3) $\{x_i, v_j\}$ $1 \le i \ne j \le n$

$\{\bar{x}_i, v_j\}$ $1 \le i \ne j \le n$.

(4) $\{\alpha, c_i\}$ if α is not in c_i
$1 \le i \le k$.

Clearly for a consistent colouring of G_φ we need n colours to colour the vertices v_i.

Assume v_i is coloured with colour i

One of $x_i, \bar{x}_j$ can be coloured with colour i.

Thus we need an extra colour, say $n+1$ to colour the other literal.

Claim G_φ is $n+1$ colourable iff φ is satisfiable.

Assume that φ is satisfiable. Let ν be a satisfying assignment

Under this assignment one of

$x_i, \bar{x}_i$ is TRUE.

Colour the TRUE literal with

colour i & the other literal with

colour $n+1$

Colour v_i with colour i.

To colour C_j, $1 \le j \le k$.

W.l.g. assume that $n \geq 4$

Since each clause contains at most 3 literals, $\exists$ a pair $\{x_i, \bar{x}_i\}$ that does not belong to C_j

C_j cannot be coloured with colour $n+1$

C_j contain a TRUE literal α which receive some colour i

Colour C_j with colour i.

Clearly this is an $n+1$-colouring of G_φ

Conversely, suppose G_φ is $n+1$-colourable

W.l.o.g. let v_i receives colour i

None of C_j can be coloured with colour $n+1$

Suppose C_j is coloured with colour i

$\therefore$ One of x_i or $\bar{x}_i$ is in C_j