Propositional Logic

P = set of atomic propositions | atoms | atomic formulas

$P = \{p_0, p_1, p_2, \ldots\}$

Well-formed formulas | propositional

Connectives : $\neg$ (Negation)

$\lor$ (OR | disjunction)

$\land$ (AND | conjunction)

$\rightarrow$ (Implication)

$\leftrightarrow$ (double implication)

The set of WFF is the smallest set ~~cons~~ containing $\mathcal{P}$ and satisfying the following

- If $\alpha \in WFF$ then $(\neg \alpha) \in WFF$
- If $\alpha, \beta \in WFF$, then so are $(\alpha \wedge \beta), (\alpha \vee \beta) \in WFF$
- If α, β are in WFF then so are $(\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta)$

Ex Show that WFF is countable

Defn A valuation | truth assignment is a f^n

$$v : P \longrightarrow \{TRUE, FALSE\} = \{1, 0\}$$

We extend v to WFF as follows:

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We extend v to WFF as follows:

(i) If $\alpha = p$ an atom

$v(\alpha) = v(p)$

(ii) If $\alpha = \neg\beta$, then

$$v(\alpha) = \begin{cases} 0 & \text{if } v(\beta) = 1 \\ 1 & \text{if } v(\beta) = 0 \end{cases}$$

(iii) If $\alpha = \beta \vee \gamma$ then

$$v(\alpha) = \begin{cases} 1 & \text{if } v(\beta) = 1 \text{ or } v(\gamma) = 1 \\ 0 & \text{o.w.} \end{cases}$$

(iv) If $\alpha = \beta \wedge \gamma$, then

$$v(\alpha) = \begin{cases} 1 & \text{if } v(\beta) = 1 \text{ \underline{and} } v(\gamma) = 1 \\ 0 & \text{o.w.} \end{cases}$$

(v) If $\alpha = \beta \rightarrow \gamma$, then

$$v(\alpha) = \begin{cases} 0 & \text{if } v(\beta) = 1 \text{ and } v(\gamma) = 0. \\ 1 & \text{o.w.} \end{cases}$$

(vi) If $\alpha := \beta \leftrightarrow \gamma$

$$v(\alpha) = \begin{cases} 1 & \text{if } v(\beta) = v(\gamma) \\ 0 & \text{o.w.} \end{cases}$$

$\underline{\text{Def}^n}$ A a set $X \subseteq WFF$ is

satisfiable if $\exists$ an assignment v

s.t. $v(\alpha) = 1 \quad \forall \alpha \in X$.

If $X = \{\alpha\}$ we say that α is satisfiable

X/α is unsatisfiable if it is $\underline{\text{not}}$ satisfiable.

α is called a Tautology / Valid

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$v(\alpha) = 1$

satisfiable

unsatisfiable

Example p, $\underline{p \vee \neg p}$, $\underline{\underline{p \wedge \neg p}}$

tautology

Proposition α is a tautology iff $\neg\alpha$ is unsatisfiable.

pf Ex.

Ex There are $n+1$ pigeons & n pigeonholes.

For $1 \leq i \leq n+1$, $1 \leq j \leq n$

p_{ij} : $i^{\underline{th}}$ pigeon is occupying the $j^{\underline{th}}$ pigeonhole.

Write down a WFF that expresses the Pigeon Hole Principle.

Logical Equivalence

Let α, β be WFFs.

α is said to be logically equivalent to β, & we write $\alpha \equiv \beta$ if for any valuation v,

$$v(\alpha) = v(\beta)$$

e.g. $\alpha \to \beta \equiv \neg\alpha \vee \beta$

$\alpha \leftrightarrow \beta \equiv (\alpha \to \beta) \wedge (\beta \to \alpha)$

Ex

(i) $\alpha \wedge \alpha \equiv \alpha$, $\alpha \vee \alpha \equiv \alpha$

(ii) $\alpha \wedge \beta \equiv \beta \wedge \alpha$, $\alpha \vee \beta \equiv \beta \vee \alpha$

(iii) $(\alpha \wedge (\beta \wedge \gamma)) \equiv (\alpha \wedge \beta) \wedge \gamma$

$\alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma$

(iv) $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

$\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

(vi) $\neg(\alpha \wedge \beta) \equiv (\neg\alpha) \vee (\neg\beta)$

$\neg(\alpha \vee \beta) \equiv (\neg\alpha) \wedge (\neg\beta)$

(viii) $\neg\neg\alpha \equiv \alpha$.