

Ex Show that

$$\alpha \equiv \beta \text{ iff } \models \alpha \leftrightarrow \beta$$

Ex Check whether each of the following

- is (a) a tautology (b) satisfiable
- (c) ~~unsatisfiable~~ unsatisfiable.

(2) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

(3) $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$

$$(iii) \quad p \wedge \neg q$$

$$(iv) \quad (p \vee \neg q) \rightarrow q$$

$$(v) \quad \neg (p \rightarrow q) \rightarrow (p \wedge \neg q)$$

Defⁿ Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n, \mathcal{A}$ be WFFs. Then \mathcal{A} is said to be a tautological consequence / logical consequence of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ if for any valuation

$$\mathcal{V}, \text{ if } \mathcal{V}(\mathcal{A}_1) = \mathcal{V}(\mathcal{A}_2) = \dots = \mathcal{V}(\mathcal{A}_n) = 1$$

then $\mathcal{V}(\mathcal{A}) = 1$
In this case we write $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vDash \mathcal{A}$

Let $X \subseteq \text{WFF}$.

Then β is a logical consequence of X
if we write $X \models \beta$ if for any
valuation v , if $v(\alpha) = 1 \ \forall \alpha \in X$

then $v(\beta) = 1$.

Then The following are equivalent
(TFAE)

$$(1) \quad \delta_1, \delta_2, \dots, \delta_n \neq \delta$$

$$(2) \quad \neg \delta_1 \wedge \neg \delta_2 \wedge \dots \wedge \neg \delta_n \rightarrow \delta$$

$$(3) \quad \delta_1 \wedge \delta_2 \wedge \dots \wedge \delta_n \wedge \neg \delta \quad \text{is}$$

unsatisfiable.

pf (1) \rightarrow (2)

Assume (1). We need to

show $F \varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n \rightarrow \varphi$

Fix a valuation v .

Suppose $v(\varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \varphi) = 0$

$\Rightarrow v(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n) = 1$ and $v(\varphi) = 0$

$\Rightarrow v(\varphi_1) = v(\varphi_2) = \dots = v(\varphi_n) = 1$ and $v(\varphi) = 0$
contradiction ①

(2) \rightarrow (3)

Assume (2)

We need to show $\exists, \neg \exists, \neg \exists, \neg \exists$
is unsatisfiable. Suppose not.

Hence \exists a valuation v s.t.

$$v(\exists, \neg \exists, \neg \exists, \neg \exists) = 1$$

$$\implies v(\exists, \neg \exists) = 1 \text{ and } v(\neg \exists) = 1$$

$$\implies v(\exists, \neg \exists) = 1 \text{ and } v(\exists) = 0$$

contradiction (2)

(3) \rightarrow (1)

Assume (2)

We need to show $\delta_1, \dots, \delta_n \neq \delta$

Fix a valuation s.t. $v(\delta_1) = \dots = v(\delta_n) = 1$

Hence $v(\delta_1 \wedge \dots \wedge \delta_n) = 1$

By (2) $v(\delta_1 \wedge \dots \wedge \delta_n \wedge \neg \delta) = 0$

Since $v(\delta_1 \wedge \dots \wedge \delta_n) = 1$, we must have

$$v(\neg \delta) = 0$$

$$\Rightarrow v(\delta) = 1$$

$$z_1, z_2, \dots, z_n \neq z$$

Ex Which of the following are correct.

(2) $p \rightarrow q, p \neq q$

(ii) $p \rightarrow q, q \neq p$

(iii) $p \rightarrow q, q \neq p$

(iv) $p \rightarrow (q \rightarrow r), (r \vee p), q \neq r \rightarrow s$

Defⁿ A formula α is said to be in Disjunctive Normal Form (DNF) if α is of the form

$$C_1 \vee C_2 \vee \dots \vee C_k, \text{ where}$$

Each "Clause" C_i is a conjunction of atoms or negation of atoms (i.e. literals)

e.g. $\underline{(p_3 \wedge \neg p_5 \wedge p_6)} \vee \underline{(p_1)} \sim \underline{(\neg p_4 \wedge p_5 \wedge \neg p_7)} \vee \underline{\neg p_8}$

By α is said to be in Conjunctive Normal Form (CNF) if α is

of the form

$$C_1 \wedge C_2 \wedge \dots \wedge C_k$$

Where each clause C_i
is a disjunction of literals.

$$e.g. (p_5 \vee \neg p_6) \wedge (\neg p_8) \wedge (p_3 \vee \neg p_7 \vee \neg p_8)$$

Thm Every propositional formula can
be converted into an equivalent
formula in DNF/CNF

Step 1 Eliminate $\rightarrow, \leftrightarrow$

(Replace $\alpha \rightarrow \beta$ by $\neg \alpha \vee \beta$.
 $\alpha \leftrightarrow \beta$ by $(\neg \alpha \vee \beta) \wedge (\neg \beta \vee \alpha)$)
 $x \vee y \wedge z \wedge w$
+

Step 2 Move \neg inwards (replacing $\neg \neg \alpha$ by α)

$$\neg(\alpha \wedge \beta) \rightsquigarrow (\neg \alpha \vee \neg \beta)$$

Step 3 Replace \wedge by $+$ and \vee by \cdot } CNF
& apply the distributive laws

$$\left((\neg p \vee (p \wedge \neg q)) \wedge (\neg \vee (\neg p \wedge q)) \right)$$

Replace \wedge by $+$ and \vee by \cdot

$$\left(\bar{p} \cdot (p + \bar{q}) \right) + \left(\neg \cdot (\bar{p} + q) \right)$$

$$\left(\bar{p} p + \bar{p} \bar{q} \right) + \left(\neg \bar{p} + \neg q \right)$$

$$\left((\bar{p} p) \wedge (\bar{p} \bar{q}) \right) \wedge \left((\neg \bar{p}) \vee (\neg q) \right)$$

Ex Convert the following into both

DNF and CNF.

a) $(p \wedge (q \vee r)) \vee (q \wedge (p \vee r))$

b) $((\neg p \vee (p \wedge r)) \wedge (r \vee (\neg p \wedge q)))$

Axiomatization of Propositional Logic Ax

The axioms of Ax are the following.

$$\underline{A1} \quad \alpha \rightarrow (\beta \rightarrow \alpha)$$

$$\underline{A2} \quad (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$$

$$\underline{A3} \quad (\neg \beta \rightarrow \neg \alpha) \rightarrow ((\neg \beta \rightarrow \alpha) \rightarrow \beta)$$

Rule of Inference (Modus Ponens)

Infer β from α and $\alpha \rightarrow \beta$

We write
$$\frac{\alpha, \alpha \rightarrow \beta}{\beta}$$

Defⁿ A wff α is said to be a theorem of A if $\vdash \alpha$ seqⁿ where $\varphi_1, \dots, \varphi_n$

$$(2) \varphi_n = \alpha$$

and (ii) Each α_i is an axiom
or α_i follows from two
previous elements by MP.

The seqⁿ $\alpha_1, \dots, \alpha_n = \alpha$ is called a
proof of α if we write $\boxed{\vdash \alpha}$

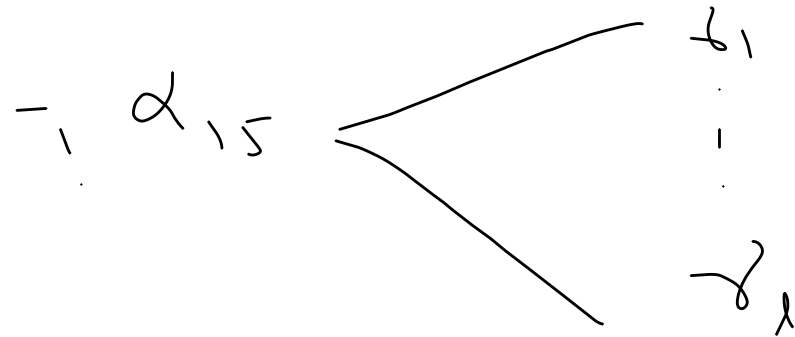
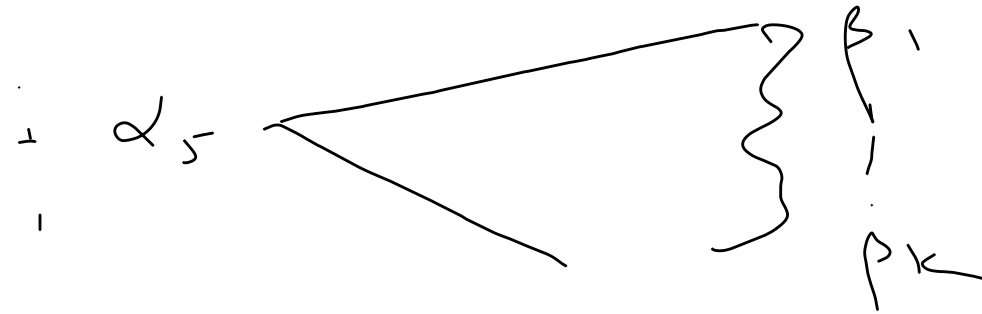
Ex (1) WFF is a theorem of $A \times$ iff
 $\exists \alpha$ sepⁿ $\alpha_1, \dots, \alpha_n$ where

$$(2) \alpha_2 = \alpha$$

and (2) α_2 is an axiom or a theorem
or follows from 2 previous
theses by MP.

α_1

α_2



$\alpha_{11} = \alpha$

Proof

$$\vdash p \rightarrow p$$

1

$$p \rightarrow ((p \rightarrow p) \rightarrow p) \quad A2$$

$$\rightarrow ((p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p))$$

2

$$p \rightarrow ((p \rightarrow p) \rightarrow p) \quad A1$$

3

$$(p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p) \quad MP$$

4

$$p \rightarrow (p \rightarrow p) \quad A1$$

Ex Show that the following
are theorems

$$\textcircled{a} (\neg \beta \rightarrow \neg \alpha) \leftrightarrow (\alpha \rightarrow \beta)$$

$$\textcircled{b} \vdash \alpha \rightarrow \neg \neg \alpha \quad \vdash \neg \neg \alpha \rightarrow \alpha$$

$$\textcircled{c} \vdash \alpha \vee \neg \alpha$$

From 3 & 4 by MP we have.

$$(p \rightarrow p)$$