Ex Show that

$$\alpha \equiv \beta \quad \text{iff} \quad \models \alpha \longleftrightarrow \beta$$

Ex Check whether each of the following is (a) a tautology (b) satisfiable (c) ~~the~~ unsatisfiable

(i) $\big(p \to (q \to r)\big) \longrightarrow \big((p \to q) \longrightarrow (p \to r)\big)$

(ii) $\big(p \to (q \to r)\big) \longleftrightarrow \big((p \wedge q) \longrightarrow r\big)$

(iii) $p \wedge \neg q$

(iv) $(p \vee q) \rightarrow q$

(v) $\neg (p \rightarrow q) \rightarrow (p \wedge \neg q)$

Def^n Let $\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma$ be WFFs. Then γ is said to be a tautological consequence | logical consequence of $\gamma_1, \gamma_2, \ldots, \gamma_n$ if for any valuation v, if $v(\gamma_1) = v(\gamma_2) = \cdots = v(\gamma_n) = \underline{1}$ then $v(\gamma) = \underline{1}$

In this case we write $\gamma_1, \gamma_2, \ldots, \gamma_n \models \gamma$

Let $X \subseteq WFF$.

Then γ is a logical consequence of X & we write $X \models \gamma$ if for any valuation v, if $v(\alpha) = 1$ $\forall \alpha \in X$, then $v(\gamma) = 1$.

Thm The following are equivalent (TFAE)

(1) $\gamma_1, \gamma_2, \ldots, \gamma_n \models \gamma$

(2) $\models \gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_n \to \gamma$

(3) $\gamma_1 \wedge \gamma_2 \ldots \wedge \gamma_n \wedge \neg \gamma$ unsatisfiable

Pf (1) $\to$ (2)

Assume (1). We need to show $\models \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \to \gamma$

Fix a valuation v.

Suppose $v(\gamma_1 \wedge \cdots \wedge \gamma_n \to \gamma) = 0$

$\Rightarrow v(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) = 1$ and $v(\gamma) = 0$

$\Rightarrow v(\gamma_1) = v(\gamma_2) = \cdots = v(\gamma_n) = 1$ and $v(\gamma) = 0$

contradiction (1)

(2) $\to$ (3)

Assume (2)

We need to show $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \wedge \neg\gamma$ is unsatisfiable. Suppose not.

Hence $\exists$ a valuation v s.t.

$$v(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \wedge \neg\gamma) = 1$$

$$\Rightarrow v(\gamma_1 \wedge \cdots \wedge \gamma_n) = 1 \text{ and } v(\neg\gamma) = 1$$

$$\Rightarrow v(\gamma_1 \wedge \cdots \wedge \gamma_n) = 1 \text{ and } v(\gamma) = 0$$

contradiction (2)

(3) $\to$ (1)

Assume (3)

We need to show $\gamma_1, \ldots, \gamma_n \models \gamma$

Fix a valuation s.t. $v(\gamma_1) = \ldots = v(\gamma_n) = 1$

Hence $v(\gamma_1 \wedge \ldots \wedge \gamma_n) = 1$

By (3) $v(\underbrace{\gamma_1 \wedge \ldots \wedge \gamma_n} \wedge \underline{\underline{\neg\gamma}}) = 0$

Since $v(\gamma_1 \wedge \ldots \wedge \gamma_n) = 1$, we must have

$v(\neg\gamma) = 0$

$\Rightarrow v(\gamma) = 1$

$\gamma_1, \gamma_2, \ldots, \gamma_n \models \gamma$

Ex Which of the following are correct.

(i) $p \to q,\ p \models q$

(ii) $p \to q,\ q \models p$

(iii) $p \to q,\ \neg q \models \neg p$

(iv) $p \to (q \to r),\ (\neg s \vee p),\ q \models s \to r$

Def^n A formula α is said to be in Disjunctive Normal Form (DNF) if α is of the form

$$C_1 \vee C_2 \vee \cdots \vee C_k \text{, where}$$

each "clause" C_i is a conjunction of atoms or negation of atoms (i.e. literals) e.g. $\underline{(p_3 \wedge \neg p_5 \wedge p_6)} \vee \underline{(p_1)} \vee \underline{(\neg p_4 \wedge p_5 \wedge \neg p_7)} \vee \underline{\neg p_8}$

Ily α is said to be in Conjuntive Normal Form (CNF) if α is of the form

$$\boxed{C_1 \wedge C_2 \wedge \cdots \wedge C_k},$$

where each clause C_i is a disjunction of literals.

e.g. $(p_5 \vee \neg p_6) \wedge (\neg p_8) \wedge (p_3 \vee \neg p_7 \vee \neg p_8)$

<u>Thm</u> Every propositional formula can be converted into an equivalent formula in DNF / CNF

Step 1 Eliminate $\rightarrow$, $\leftrightarrow$

(Replace $\alpha \rightarrow \beta$ by $\neg\alpha \vee \beta$.
$\alpha \leftrightarrow \beta$ by $(\neg\alpha \vee \beta) \wedge (\neg\beta \vee \alpha)$)

$x.y.z + x'.y.z.w'$

$+$

Step 2 Move $\neg$ inwards (replacing $\neg\neg\alpha$ by α)

$\neg(\alpha \wedge \beta) \rightsquigarrow (\neg\alpha \vee \neg\beta)$

Step 3 Replace $\wedge$ by $+$ and $\vee$ by $\cdot$ & apply the distributive laws } CNF

$$((\neg p \vee (p \wedge \neg q)) \wedge (r \vee (\neg p \wedge q)))$$

Replace $\wedge$ by $\cdot$ and $\vee$ by $+$

$$(\bar{p} \cdot (p + \bar{q})) + (r \cdot (\bar{p} + q)).$$

$$(\bar{p} \cdot p + \bar{p} \cdot \bar{q}) + (r \cdot \bar{p} + r \cdot q).$$

$$(\underline{(\bar{p} \vee p)} \wedge \underline{(\bar{p} \vee \bar{q})}) \wedge (\underline{(r \vee \bar{p})} \wedge \underline{(r \vee q)})).$$

Ex Convert the following into both DNF and CNF.

(a) $(p \wedge (q \vee r)) \vee (q \wedge (p \vee r))$

(b) $((\neg p \vee (p \wedge \neg q)) \wedge (r \vee (\neg p \wedge q)))$

Axiomatization of Proposition Logic $\underline{Ax}$

The axioms of Ax are the following

$\underline{A1}$ $\quad \alpha \to (\beta \to \alpha)$

$\underline{A2}$ $\quad (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

$\underline{A3}$ $\quad (\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$

Rule of Inference (Modus Ponens)

Infer β from α and $\alpha \to \beta$

We write $\dfrac{\alpha,\ \alpha \to \beta}{\beta}$

Def^n A wff α is said to be a theorem of Ax if $\exists$ a seq^n $\alpha_1, \ldots, \alpha_n$ where

(2) $\alpha_n = \alpha$

and (ii) each α_i is an axiom

or α_i follows from two

previous elements by MP.

The seq$^{\underline{n}}$ $\alpha_1 \ldots \alpha_n = \alpha$ is called a

proof of α & we write $\boxed{\vdash \alpha}$

Ex A WFF is a theorem of Ax iff $\exists$ a seq^n $\alpha_1, \ldots, \alpha_n$ where

(i) $\alpha_n = \alpha$

and (ii) α_i is an axiom or a theorem or follows from 2 previous terms by MP.

α_1

α_2

$\vdots$

α_5 $\left\{ \begin{array}{l} \beta_1 \\ \vdots \\ \beta_k \end{array} \right.$

$\vdots$

α_{15} $\begin{array}{l} \gamma_1 \\ \vdots \\ \gamma_l \end{array}$

$\alpha_n = \alpha$.

Prop. $\vdash p \to p$.

1. $p \to ((p \to p) \to p) \to ((p \to (p \to p)) \to (p \to p))$ A2

2. $p \to ((p \to p) \to p)$ A1

3. $(p \to (p \to p)) \to (p \to p)$ MP

4. $p \to (p \to p)$ A1

Ex Show that the following are theorems

(a) $(\neg\beta \to \neg\alpha) \longleftrightarrow (\alpha \to \beta)$.

(b) $\vdash \alpha \to \neg\neg\alpha$. $\vdash \neg\neg\alpha \to \alpha$.

(c) $\vdash \alpha \vee \neg\alpha$.

From 3 & 4 by MP we have.

$$(p \rightarrow p)$$