$\mathcal{PL}$.

A1 $\quad ((\alpha \to \beta) \to \alpha)$.

(A2) $\quad (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

A3 $\quad ((\neg\beta \to \neg\alpha) \to (\neg\beta \to \alpha)) \to \beta$

Rule of Inference: Modus Ponens $\quad \dfrac{\alpha,\ \alpha \to \beta}{\beta}$

Theorems

Ex $\vdash \alpha \to \neg\neg\alpha$; $\vdash \neg\neg\alpha \to \alpha$.

$\vdash \alpha \vee \neg\alpha$.

$\vdash (\neg\beta \to \neg\alpha) \leftrightarrow (\alpha \to \beta)$

Lemma (Soundness Theorem)

For any formula α,
if $\vdash \alpha$ then $\models \alpha$.

<u>Pf.</u> Suppose $\vdash \alpha$. Let
$\alpha_1, \alpha_2, \ldots, \alpha_r = \alpha$ be a proof of α.
We shall prove by induction on i
~~the~~ that $\models \alpha_i$

Since α_1 is either A1 or A2 or A_3, it follows that $\vdash \alpha$

(Ex)

Assume that $\vdash \alpha_i$ for all $i < n$.

If α_n is an axiom then $\vdash \alpha_n$

So assume that α_n follows from α_i, α_j $(i, j < n)$ by MP.

Hence one of them, say α_j, is of the form $\alpha_i \to \alpha_n$.

By induction hyp.,

$\models \alpha_i$ and $\models \alpha_i \to \alpha_n$.

$\therefore \models \alpha_n$. //.

Defn A formula α is said to be **consistent** if $\nvdash \neg\alpha$.

A finite set of formulas $X = \{\alpha_1, \ldots, \alpha_n\}$ is consistent if $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$ is consistent.

A set $X \subseteq WFF$ is consistent if every finite subset of X is consistent

Def^n A set $X \subseteq$ WFF is called a maximal consistent set (MCS) if (i) X is consistent & (ii) $\forall \beta \notin X$, $X \cup \{\beta\}$ is not consistent

Lemma 1. (Lindenbaum)

Every consistent set of formulas X can be extended to an MCS.

Pf We shall construct a seq^n $\{X_n\}$ as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be an enumeration of the formulas of PL

$$X_0 = X.$$

$$X_{i+1} = \begin{cases} X_i \cup \{\alpha_i\} & \text{if } X_i \cup \{\alpha_i\} \text{ is consistent} \\ X_i & \text{o.w.} \end{cases}$$

Let $Y = \bigcup_{i \geq 0} X_i$

Clearly, $Y \supseteq X$.

By construction each X_i is consistent

<u>Claim</u> Y is a maximal consistent set

Let A be a finite subset of Y.

Clearly, $A \subseteq X_i$ for some i.

Since X_i is consistent A is also consistent.

Let $\beta \notin Y$.

Let $\beta = \alpha_j$ for some j.

Since $\alpha_j \notin Y$,

$\alpha_j \notin X_{j+1}$

Since α_j has not been added at step $j+1$ of our construction,

$X_j \cup \{\alpha_j\}$ is <u>not</u> consistent

Hence $\exists$ a finite set $F \subseteq X_j$ s.t.

$F \cup \{\alpha_j\}$ is not consistent.

Clearly, $F \cup \{\alpha_j\} \underset{\neq}{\subseteq} Y \cup \{\beta\}$

$\Longrightarrow Y \cup \{\beta\}$ is not consistent.

<u>Lemma 2</u> Let X be an MCS.

(i) For any formula α, $\alpha \in X$ iff $\neg\alpha \notin X$.

(ii) $\alpha \vee \beta \in X$ then $\alpha \in X$ or $\beta \in X$.

Defn For every MCS X, we define a valuation v_X as follows:

For any $p \in P$, $v_X(p) = 1$ iff $p \in X$.

Prop. Let v_X be as above. Then for any formula α,

$$v_X(\alpha) = 1 \text{ iff } \alpha \in X.$$

Pf By induction on $|\alpha|$

Base case α is atomic.

Then $v_X(\alpha) = 1$ iff $\alpha \in X$.

Suppose $\alpha = \neg\beta$.

$v_X(\alpha) = 1$ iff $v_X(\beta) = 0$

iff $\beta \notin X$ iff $\neg\beta \in X$ by $\lambda 2$

Suppose $\alpha = \beta \vee \gamma$.

$v_X(\alpha) = 1$ iff $v_X(\beta) = 1$ or $v_X(\gamma) = 1$.

iff $\beta \in X$ or $\gamma \in X$

iff $\beta \vee \gamma \in X$ by $\alpha 2$

Lemma 3 (Henkin)
If α is consistent, then α is satisfiable.

<u>Pf.</u> $\{\alpha\}$ is consistent.
By L1, $\exists$ an MCS X s.t.
$\alpha \in X$. Clearly $v_X(\alpha) = 1$.

Completeness Thm.

For any formula α, if $\models \alpha$ then $\vdash \alpha$.

Pf Suppose $\nvdash \alpha$.

We have $\vdash \neg\neg\alpha \to \alpha$

If $\vdash \neg\neg\alpha$, then by MP $\vdash \alpha$
Contradiction.

Hence $\nvdash \neg\neg\alpha$. By def^n $\neg\alpha$ is consistent.

By L3, for some valuation v, $v(\neg\alpha) = 1$

$\Rightarrow v(\alpha) = 0 \Rightarrow \nvDash \alpha$