$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

<u>Prop</u>. Let $A_1, A_2, \ldots, A_n$ be a seq$^{\underline{n}}$ of events over Ω s.t. $P(A_i) > 0 \ \forall i$.

$$P(A_1 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1}),$$

$\boxed{\text{provided } P(A_1 \cap \ldots \cap A_{n-1}) > 0}$

Pf By induction on n.

For $n=2$, it follows from def^n.

$$P(A_1 \cap \cdots \cap A_{n-1} \cap A_n)$$

$$= P(A_1 \cap \cdots \cap A_{n-1}) \cdot P(A_n \mid A_1 \cap \cdots \cap A_{n-1})$$

$$= P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_{n-1} \mid A_1 \cap \cdots \cap A_{n-2})$$

$$\cdot P(A_n \mid A_1 \cap \cdots \cap A_n)$$

Ex Suppose an Urn contains

5 W, 4 R and 3 B chips.

Four chips are selected sequentially.

What is the prob. of obtaining

(W, B, W, R)

Sol^n : A : Selection a W in the 1st draw

B : " a B . . . 2nd draw

C : selecting a W . . . 3rd draw

D : sel. . . R . . . 4th draw

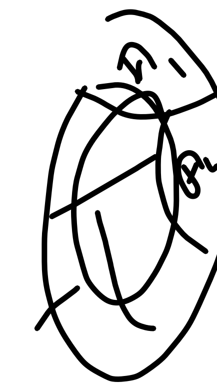
$$P(A \cap B \cap C \cap D) = P(A) \cdot P(B|A)\, P(C|A \cap B) \cdot P(D|A \cap B \cap C)$$

$P(A) = 5/12$, $P(B|A) = \frac{4}{11}$

$P(C|A \cap B) = 4/10$. $P(D|A \cap B \cap C) = 3/9$.

Bayes' Thm.

Let $A_1, A_2, \dots, A_n$ be a seq^n of mutually exclusive & exhaustive events s.t. $P(A_i) > 0 \ \forall i$.



Then for any event B s.t. $P(B) > 0$.

$$P(A_i|B) = \frac{P(A_i)\cdot P(B|A_i)}{\sum_{i=1}^{n} P(A_i)\cdot P(B|A_i)}$$

Pf

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

$$= \frac{P(A_i) \cdot P(B \mid A_i)}{P(B)}$$

$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B \mid A_i)$$

Ex Urn I contains 3W & 4R
Urn II contains 6W & 3R chips

A biased coin, twice as likely to show heads as tails, is tossed once.
If H appears, draw a chip from Urn I.
If T appears, draw a chip from Urn II

If the chip drawn is W
what is the prob. that the coin
showed tails?

B : a W chip is drawn

A_1 : the coin showed tails.

A_2 : " " " " heads.

W have to find

$$P(A_1 \mid B) = \frac{P(A_1) \cdot P(B \mid A_1)}{P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2)}$$

$$= \frac{\frac{1}{3} \cdot \frac{6}{9}}{\frac{1}{3} \cdot \frac{6}{9} + \frac{2}{3} \cdot \frac{3}{7}}$$

Ex Urn I : 2 R & 4 W chips.

Urn II : 3 R & 1 W chips

A chip is drawn at random from Urn I & transferred to urn II. A chip is drawn at random from Urn II. What is the prob. of getting a R chip?

Sol^n

B : A red chip is drawn.

A_1 : A W was transferred.

A_2 : A R chip was transferred.

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)$$

$$= 4/6 \cdot 3/5 + \frac{2}{6} \cdot \frac{4}{5}$$

Independent Events

Two events A and B are said to be independent if $\boxed{P(A\cap B) = P(A)\cdot P(B)}$

$$P(A\cap B) = P(B)\cdot P(A|B)$$

$$\cancel{P(B)}\cdot P(A|B) = P(A)\cdot \cancel{P(B)}$$

$$P(A) = P(A|B)$$

Suppose A and B are independent.

$$P(A) = P(B) \cdot P(A \mid B) + P(B^c) \cdot P(A \mid B^c).$$

$$= P(B) \cdot P(A) + P(B^c) \cdot P(A \mid B^c)$$

$$P(A)\cancel{\left[1 - P(B)\right]} = \cancel{P(B^c)} \cdot P(A \mid B^c)$$

$$\Rightarrow P(A) = P(A \mid B^c)$$

IIly one can prove

$$P(B) = P(B|A)$$

Ex. If A & B are independent then show that A^c & B^c are also independent events.

Ex A card is drawn from a deck of cards. Let

A : event of drawing a King.

B : event of drawing a Club.

$A \cap B$: event of drawing a King of clubs

$P(A \cap B) = \frac{1}{52}$. $P(A) = \frac{4}{52}$. $P(B) = \frac{13}{52} = \frac{1}{4}$

$P(A \cap B) = P(A) \cdot P(B)$

Ex Let $\Omega = \{s_1, s_2, s_3, s_4\}$.

$P(\{s_i\}) = \frac{1}{4}$; $i = 1, 2, 3, 4$.

$A_1 = \{s_1, s_2\}$; $A_2 = \{s_1, s_3\}$, $A_3 = \{s_1, s_4\}$.

$P(A_1) = \frac{1}{2} = P(A_2)$.

$P(A_1 \cap A_2) = \frac{1}{4}$

$$\left.\begin{array}{l} P(A_1 \cap A_2) = P(A_1) \cdot P(A_2) \\ P(A_1 \cap A_3) = P(A_1) P(A_3) \end{array}\right\}$$

$P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$

$P(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \neq P(A_1) \cdot P(A_2) \cdot P(A_3)$

Def^n A seq^n $A_1, A_2, \ldots, A_n$ of events is said to be <u>mutually independent</u> if for any seq^n $1 \le i_1 < i_2 < \ldots < i_k \le n$

$$P\left(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}\right) = P(A_{i_1}) \cdot P(A_{i_2}) \ldots P(A_{i_k})$$

Def^n: Consider an experiment in which there are exactly two outcomes viz.

1/S : success

0/F : failure.

Assume $P(S) = p$

$P(F) = 1 - p \overset{\text{Def}^n}{=} q$.

Consider a seq^n of n trials s.t. the outcome of the i^{th} trial does not depend on the outcomes of the previous trials.

We wish to calculate the prob. of k successes in a seq^n of n independent Bernoulli trials.

$$F\,F \,.\,.\,.\, S\,F \,.\,.\, S \,.\,.\,.\,.\, \qquad .\,.\,.\, S \,.\,.\,.$$

$$P(k \text{ success}) = p^{k} \cdot q^{n-k}.$$

$P($ k successes in n Bernoulli trials

$$= \boxed{\binom{n}{k} p^k q^{n-k}}$$