Half-Yearly Presentation

Lawande Shital Sanjay

IAI TCG CREST and RKMVERI

August 26, 2022

Courses completed semester-wise and marks obtained

Semester-1:

1.Algebra and its Applications-70
2.Introduction to Stochastic Process-77
3.Discrete Mathematics-55
4.Cryptology and Security-52

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1.Algebra and its Applications-70

2.Introduction to Stochastic Process-77

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4.Cryptology and Security-52

Semester-2:

1.Trends in Combinatorial Topology-64

2. Introduction to Statistics and Probability-57

3. Knot Theory-74

4.Research Methodology-76

My Co-Guide and Area of Work

- Guide - Prof. Sukumar Das Adhikari
- Co guide - Kuldeep Saha.

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- Co guide - Kuldeep Saha.
- Area of Work - Topology.

Papers and books read

- Algebraic Topology by Allen Hatcher.
- Knots and Links by Dale Rolfsen.
- A Primer on Mapping Class Groups by Benson Farb and Dan Margalit.
- A research paper on Computing Persistent Homology by Afra Zomorodian and Gunnar Carlsson

Mapping Class Group

- Definition of Mapping class group.

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Mapping class group: Mod(S)

Group of isotopy classes of elements of $Homeo^+(S, \partial S)$, where isotopies are required to fix boundary pointwise.
i.e Group of orientation preserving homeomorphisms of S modulo isotopy.
Notations: Mod(S), MCG(S), Map(S),etc.

Mapping Class Group of disk

Mapping class group of closed disk D^2: (Alexander Lemma)

$Mod(D^2) \cong \{1\}$

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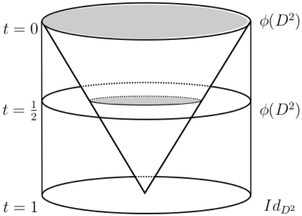

Figure: 1.Pictorial representation of the isotopy

Mapping Class Group of disk Contd.

- Define $F : D^2 \times [0,1] \to D^2$ as,

$$F(x,t) = \begin{cases} (1-t)\phi(\frac{x}{1-t}), & 0 \leqslant |x| < 1-t \\ x, & 1-t \leqslant |x| < 1. \end{cases}$$

and $F(x,1) = x, \forall x$. This gives an isotopy F from ϕ to identity.

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- Since S^2 is simply connected, $\phi(\alpha)$ is isotopic to α, so upto isotopy, ϕ fixes α pointwise.

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- Let $(D_1, \partial D_1 = \alpha)$ and $(D_2, \partial D_2 = \alpha)$ be two disks obtained by cutting S^2 along α.

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- Let $(D_1, \partial D_1 = \alpha)$ and $(D_2, \partial D_2 = \alpha)$ be two disks obtained by cutting S^2 along α.
- ϕ induces homeomorphisms $\overline{\phi}_1 : D_1 \to D_1$ and $\overline{\phi}_2 : D_2 \to D_2$ fixing α, i.e, boundary of D_1 and D_2 pointwise, $\overline{\phi}_1, \overline{\phi}_2 \in Mod(D^2)$.
- $\overline{\phi}_1, \overline{\phi}_2$ are isotopic to identity of D_1 and D_2 respectively.

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- ϕ is isotopic to identity of S^2.

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$Mod(A) \cong \mathbb{Z}$.

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$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{\phi}} & \tilde{A} \\ P \downarrow & & \downarrow P \\ A & \xrightarrow[\phi]{} & A \end{array}$$

Mapping Class Group of Annulus contd.

- $\widetilde{\phi}_1 := \widetilde{\phi}|_{\mathbb{R}\times\{1\}} : \mathbb{R} \to \mathbb{R}$. Since $\widetilde{\phi}_1$ is lift of identity map on $S^1 \times 1$, $\widetilde{\phi}_1(x) = x + n$.

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- Construction of a map $\rho : Mod(A) \to \mathbb{Z}$.

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- Let δ be an oriented simple proper arc as shown in fig.

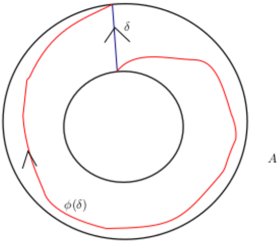

Figure: 2

Mapping Class Group of Annulus contd.

- The concatenation $\phi(\delta) * \delta^{-1}$ is a loop based at $\delta(0)$, and we define $\rho(f) = [\phi(\delta) * \delta^{-1}] = \widetilde{\phi(\delta) * \delta^{-1}}(1) \in \pi(A, \delta(0)) \cong \mathbb{Z}$.

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- Let $\tilde{\delta}$ be the unique lift of δ starting from origin then $\widetilde{\phi}(\widetilde{\delta})$ is lift of $\phi(\delta)$ with $\widetilde{\phi}(\tilde{\delta})(0) = (0, 0) = \widetilde{\phi(\delta)}(0)$.

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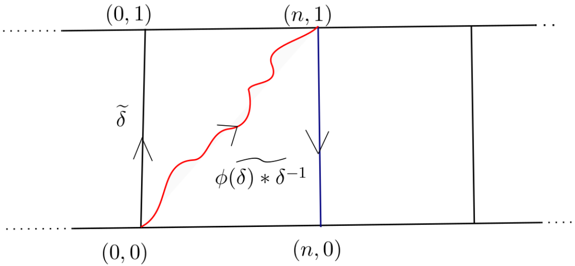

Figure: 3. Lift of loop in A

Mapping Class Group of Annulus contd.

- From fig. above we can write $\rho(f) = [\phi(\delta) * \delta^{-1}] = \widetilde{\phi(\delta)}(1)$.
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- To prove that ρ is homomorphism:
- Let $f, g \in Mod(A)$ and ϕ, ψ be the representatives of f, g and $\tilde{\phi}$,$\tilde{\psi}$ be preferred lifts of ϕ, ψ respectively.

Mapping Class Group of Annulus contd.

- Since $P \circ (\widetilde{\psi} \circ \widetilde{\phi})(\widetilde{\delta}) = \psi \circ \phi(\delta)$, $\widetilde{\psi} \circ \widetilde{\phi}(\widetilde{\delta})$ is lift of $\psi \circ \phi(\delta)$ and by unique lifting property, $\widetilde{\psi} \circ \widetilde{\phi}(\widetilde{\delta})(1) = \widetilde{\psi \circ \phi}(\delta)(1) = \widetilde{\psi \circ \phi}(\tilde{\delta})(1)$

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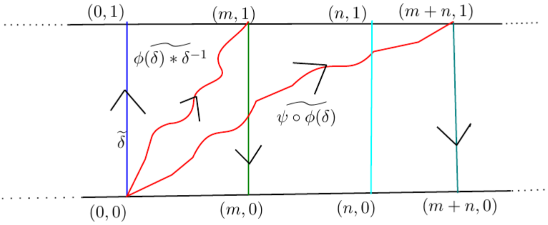

Figure: 4.Lifts of loops in A

- This gives, $\widetilde{\psi \circ \phi(\delta)}(1) = \widetilde{\phi(\delta)}(1) + \widetilde{\psi(\delta)}(1)$, so ρ is homomorphism.

Mapping Class Group of Annulus contd.

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Mapping Class Group of Annulus contd.

- To show that ρ is injective:
- Let $f \in Mod(A)$ be such that $\rho(f) = 0$ and ϕ be a homeomorphism of A representing f then $\widetilde{\phi(\delta) * \delta^{-1}}(1) = (0,0)$ which gives $\widetilde{\phi(\delta)}(1) = (0,1)$ and $\widetilde{\delta}(1) = (0,1)$.

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- Let $(D, \partial D = \delta \cup \partial A)$ be the disk obtained from A by cutting along δ.

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- Let $(D, \partial D = \delta \cup \partial A)$ be the disk obtained from A by cutting along δ.
- ϕ induces a homeomorphism $\overline{\phi}$ of D which represents an element f of $Mod(D) \simeq 1$.
- $\overline{\phi}$ is isotopic to identity map of D and hence ϕ is isotopic to identity.

Mapping Class Group of Annulus contd.

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- For any $n \in \mathbb{Z}$, linear transformation of $\mathbb{R}^2$ given by $M = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ preserves $\mathbb{R} \times [0, 1]$.

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- M defines a covering homeomorphism $\widetilde{\phi} : \widetilde{A} \to \widetilde{A}$ such that $\widetilde{\phi}|_{\mathbb{R}\times\{1\}}$ is integer translation by n.

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- So $\widetilde{\phi(\delta)}(1) = (n, 1)$ and $[\phi(\delta) * \delta^{-1}] = n$ i.e $\rho(f) = n$

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Proof:

- Let $f \in Mod(T^2)$ and $\phi : T^2 \to T^2$ be a homeomorphism representing f then ϕ induces a map $\phi_* : H_1(T^2, \mathbb{Z}) \to H_1(T^2, \mathbb{Z})$

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- $\phi \to \phi_*$ gives a homomorphism $Mod(T^2) \to Aut(\mathbb{Z}^2) \cong GL(2, \mathbb{Z})$
- Let M and L be two oriented curves on T^2 with homology clases a and b as shown in fig. then algebraic intersection number, $\hat{i}(a, b) = 1$.

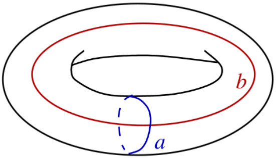

Figure: 5.Torus

Mapping Class Group of Torus T^2 contd.

- Since $H_1(T^2, \mathbb{Z}) =< a, b >$, so $\phi_*(a) = c_1 a + c_2 b$ and $\phi_*(b) = d_1 a + d_2 b$ for $c_1, c_2, d_1, d_2 \in \mathbb{Z}$.
- Orientation-preserving homeomorphisms preserves algebraic intersection number, hence $\hat{i}(a, b) = \hat{i}(\phi_*(a), \phi_*(b)) = c_1 d_2 - c_2 d_1$.

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$$[\phi_*] = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} \hat{i}(\phi_*(a), b) & \hat{i}(\phi_*(b), b) \\ \hat{i}(-\phi_*(a), a) & \hat{i}(-\phi_*(b), a) \end{bmatrix} \in SL(2, \mathbb{Z}).$$

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- This gives a homomrphism $\sigma : Mod(T^2) \rightarrow SL(2, \mathbb{Z})$, $\sigma(f) = [\phi_*]$

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- Two basic self homeomorphisms of T^2, i.e, dehn twists along longitude and meridian of the torus, $\tau_L((e^{is}, e^{it})) = (e^{i(s+t)}, e^{it})$ and $\tau_M((e^{is}, e^{it})) = (e^{is}, e^{i(s+t)})$.

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  $[\phi_*] = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} \hat{i}(\phi_*(a), b) & \hat{i}(\phi_*(b), b) \\ \hat{i}(-\phi_*(a), a) & \hat{i}(-\phi_*(b), a) \end{bmatrix} \in SL(2, \mathbb{Z})$.
- This gives a homomrphism $\sigma : Mod(T^2) \to SL(2, \mathbb{Z})$, $\sigma(f) = [\phi_*]$
- To prove that σ is surjective:
- Two basic self homeomorphisms of T^2, i.e, dehn twists along longitude and meridian of the torus, $\tau_L((e^{is}, e^{it})) = (e^{i(s+t)}, e^{it})$ and $\tau_M((e^{is}, e^{it})) = (e^{is}, e^{i(s+t)})$.

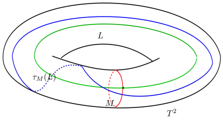

Figure: 6. Dehn twist along M

Mapping Class Group of Torus T^2 contd.

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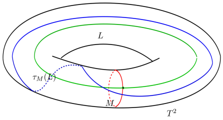

Figure: 6. Dehn twist along M

Mapping Class Group of Torus T^2 contd.

- τ_L and τ_M induces automorphisms τ_{L*} and τ_{M*} on $H_1(T^2, \mathbb{Z})$ with $[\tau_{L*}] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $[\tau_{M*}] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- $SL(2, \mathbb{Z})$ is generated by $[\tau_{L*}]$ and $[\tau_{M*}]$,so for any $M \in SL(2, \mathbb{Z})$ there is a homeomorphism say h which is compostion of powers of dehn twists along logitude and meridian of T^2 such that $\sigma(h) = M$
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- Let $f \in Mod(T^2)$, ϕ be the homeomorphism of T^2 representing f and suppose $\sigma(f) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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- Let α and β be the simple closed curves corresponding to the elements (1, 0) and (0, 1) of $\pi_1(T^2)$, then $\phi(\alpha)$ is homotopic to α and $\phi(\beta)$ is homotopic to β.

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  Lemma-1 Let α and β be two essential simple closed curves in a surface S. Then α is isotopic to β if and only if α is homotopic to β.

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Mapping Class Group of Torus T^2 contd.

Lemma-2 Let S be any surface. If $F : S^1 \times I \rightarrow$ S is a smooth isotopy of simple closed curves, then there is an isotopy H : S $\times$ I $\rightarrow$S so that$H|_{S\times 0}$ is the identity and H$|_{F(S^1\times 0)\times I} = F$.

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- $\overline{\phi}$ is isotopic to identity of A via an isotopy fixing ∂ A pointwise and hence ϕ is isotopic to identity.

References

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Thank You.