Experiment

Def^n It is a procedure that

(i) has a definite outcome

& (ii) can be repeated, theoretically, infinitely many time

Sample space: collection of outcomes of an experiment

~~Throw~~ Throwing a coin 3 times

HHH, HHT, HTH, - - , TTT

$$\Omega = \{HHH, - \quad - , TTT\}.$$

A element $s \in \Omega$ is called an sample pt.

~~Def.~~ An event A is a subset of Ω.

Ω : a finite sample space.

$\Omega = \{ H, TH, TTH, \ldots \}$

$\Omega = \{ s_1, s_2, \ldots, s_n \}$.

With each s_i we associate a real $p_i \in [0,1]$ s.t. $\sum_{i=1}^{n} p_i = 1$.

For any event $A \subseteq \Omega$,

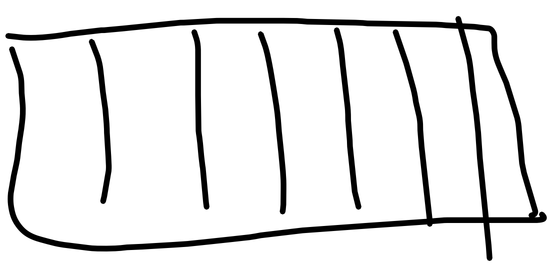
$$P(A) = \sum_{s_i \in A} p_i .$$

If $p_i = \frac{1}{n}$, then we say that the outcomes are equally likely.

In that case

$$P(A) = \frac{|A|}{n} = \frac{\#\text{ of outcomes favourable to } A}{\text{Total }\#\text{ of outcomes}}$$

Ex 1. In a randomized field experiment, a rectangular plot of land is divided into k strips of equal size. k varieties – that include A and B – are randomly drawn on these strips. What is the probability that there are r varieties between A and B,



Soln Assume that k specified varieties are grown between A and B.

$$2 \times r!\,(k-r-2)!$$

$$\binom{k}{r} \times 2 \times r! \times (k-r-2)!$$

$$P(A) = \frac{\binom{k}{r} \times 2 \times r! \times (k-r-2)!}{k!}$$

Ex2 $\boxed{k}$ identical balls are placed in $\boxed{n \text{ cells}}$. What is the prob. that a specified cell contains r balls?

Soln

$\lfloor r \text{ balls} \rfloor$

identical balls

$k - r$ balls are to be distributed

among $n-1$ cells.

$k-r+n-2$ identical balls are placed in ~~...~~ n cells.

$\binom{\cancel{\ldots}}{n-2}$ ← $O \mid \cdot - \mid \quad \mid O \mid \quad \cdot \quad O$

Ex3 n dice are thrown at a time. What is the prob. of obtaining the sum s of pts. on the dice?

Total outcomes $= 6^n$.

$$\boxed{x_1 + x_2 + \cdots + x_n = s}, \quad 1 \le x_i \le 6,$$

$$\left(x^1 + x^2 + \cdots + x^6\right)^n$$

$$\underbrace{(x^1 + x^2 + \cdots + x^6)}\times\underbrace{(x^1+x^2+\cdots+x^6)}$$

$$\cdots \times (x^1 + x^2 + \cdots + x^6)$$

Coeff^n of x^5 in $(x+\cdots+x^6)^n$ is just the # of sol^{ns} of the eq^n

$x_1+x_2+\cdots+x_n=5$, where $1\le x_i\le 6$.

Axiomatic Def^{n} of Prob. (Kolmogorov)

Ω:

Axiom 1 For any event A, $P(A) \geq 0$.

Axiom 2 $P(\Omega) = 1$. Ex Generalize.

Axiom 3 (i) If Ω is finite, then for any 2 mutually exclusive events A and B, $P(A \cup B) = P(A) + P(B)$.

(ii) If Ω is infinite, then for any seq^n of mutually exclusive events $A_1, A_2, \dots\dots$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

<u>Ex</u> Show that $\sum P(A_i)$ is convergent.

Proposition: (i) For any event A,

$$P(A^c) = 1 - P(A)$$

Consequently, $P(\phi) = 0$.

(ii) $A \subseteq B$, then $P(A) \leq P(B)$.

$$P(B) = P(B-A) + P(A) \geq P(A)$$

Consequently, for any event A,

$$P(A) \leq 1$$

Properties of the prob. fn.

Thm (i) For any 2 events A, B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(ii) If $A_i \uparrow$ $[A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots]$

Then $P(\lim A_i) = \lim P(A_i)$

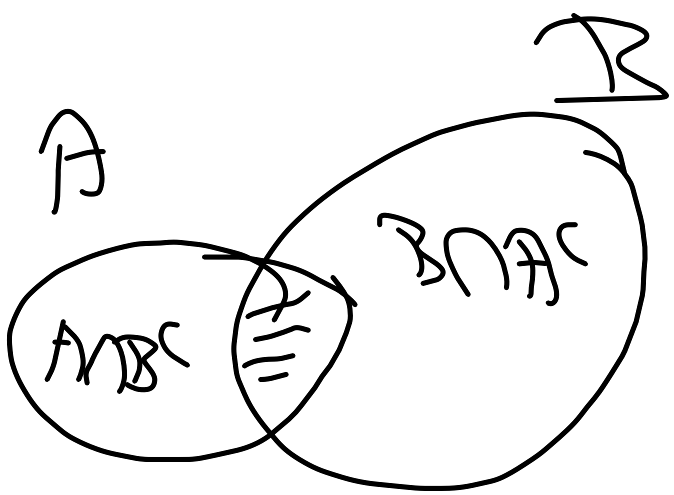
(iii) For events $A_1, A_2, \ldots, A_k$,

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Pf:

$$P(A) = P(A\cap B) + P(A \cap B^c)$$
$$P(B) = P(B\cap A) + P(B \cap A^c)$$

$$P(A)+P(B) = \left[P(A\cap B) + P(A\cap B^c) + P(B\cap A^c)\right] + P(A\cap B)$$



$$= P\left[(A\cap B) \cup (A\cap B^c) \cup (B\cap A^c)\right] + P(A\cap B)$$

$$= P(A\cup B) + P(A\cap B)$$

Pf of (ii) Set $A_0 = \phi$

$$\lim A_i = \bigcup_{i=0}^{\infty} A_i$$

$$A_n' = A_n - \bigcup_{i=1}^{n-1} A_i$$

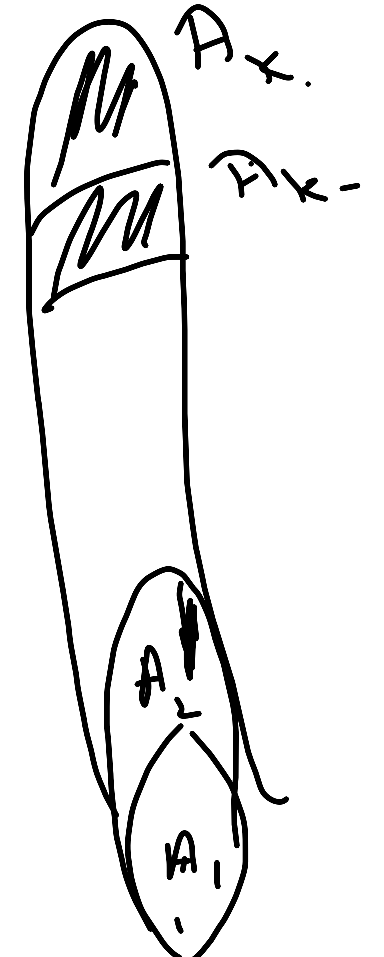
$$\lim A_i = \bigcup_{n=0}^{\infty} A_n'$$

$$P(\lim A_i) = P\left(\bigcup_{n=0}^{\infty} A_n'\right) = \sum_{n=0}^{\infty} P(A_n')$$

$$= \lim_{k\to\infty} \sum_{i=0}^{k} P(A_i')$$

$$= \lim_{k\to\infty} P\left(\bigcup_{i=1}^{k} A_i'\right)$$

$$= \lim_{k\to\infty} P(A_k).$$



(iii) $$P\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} P(A_i)$$

By induction on k.

For $k=2$,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$
$$\leq P(A_1) + P(A_2)$$

$$P(A_1 \cup A_2 \cup \dots \cup A_n)$$
$$\leq P(A_1 \cup \dots \cup A_{n-1}) + P(A_n)$$
$$\leq \underbrace{P(A_1) + P(A_2) + \dots + P(A_{n-1})}_{\text{by ind. hyp}} + P(A_n)$$

Conditional Prob.

Defn Let A, B be two events s.t. $P(B) > 0$.
We define the prob. of event A, given that B has occured to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\boxed{P(A \cap B) = P(A|B)\,P(B)}$$

Ex. A card is randomly picked from a deck of cards. What is the prob. of obtaining a club given that the card is a king.

C: be the event that the card is a club.
K: be the event that the card is a king

We want

$$P(C \mid K) = \frac{P(C \cap K)}{P(K)} = \frac{1/52}{4/52}$$

$$= \frac{1}{4}$$

Ex 2 Consider the set of families having 2 children. Assume that the 4 possible birth seq^n — (younger B, older B); (younger B, older G); (younger G, older B); (younger G, older G) — are equally likely.

What is the prob. that both children are boys given that at least one child is a boy?

A : both children are boys.

B : at least one child is a boy.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

$$= \frac{1/4}{3/4} = \frac{1}{3}$$

Thm $P(\ \cdot \mid B)$ is a prob. fn. on Ω.

Pf. (i) $P(A \mid B) \geq 0$.

(ii) $P(\Omega \mid B) = \dfrac{P(\Omega \cap B)}{P(B)} = 1$

(iii) Let a seqn $\{A_i\}$ of mutually exclusive events.

$$P\left(\bigcup_{i=1}^{\infty} A_i \,\middle|\, B\right)$$

$$= P\left(\bigcup_{i=1}^{\infty} A_i \cap B\right) \Big/ P(B)$$

$$= \sum_{i=1}^{\infty} P(A_i \cap B) \Big/ P(B)$$

$$= \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i \mid B)$$

Thm Let $\{B_i\}$ be a seq^n of mutually exclusive events s.t. $\bigcup_{i\geq 1} B_i = B$

Then $P(A\cap B) = \sum_{i=1}^{\infty} P(B_i)P(A|B_i)$

Pf $P(A\cap B) = P\left(\bigcup_{i=1}^{\infty} A\cap B_i\right)$

$= \sum_{i=1}^{\infty} P(A\cap B_i) = \sum_{i=1}^{\infty} P(B_i)\cdot P(A|B_i)$

Cor 1. If $\{B_i\}$ is a partition of Ω.

Then

$$P(A) = \sum_{i=1}^{\infty} P(B_i) P(A \mid B_i)$$

Cor 2

$$P(A) = P(B) P(A \mid B) + P(B^c) P(A \mid B^c)$$