Cobordism Classification of Manifolds

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July 25, 2022

Overview

Introduction

What are Smooth Manifolds?

They are Topological Spaces where one can do Calculus.

The Classification problem of Manifold

- One of the most concrete classification problems known so far is the **Cobordism Classification of Manifolds.**
- Introduced by R. Thom in 1954 and also described the classification problem completely.

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Introducing Smooth Manifolds

Preleminaries

Topological Manifold of dimension d

Suppose M is a topological space. We say M is Topological d-Manifold if it has the following properties:

- M is Hausdorff & 2^{nd} countable.
- M is locally Euclidean: Every point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^d$.

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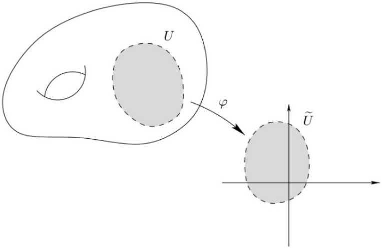
Coordinate Chart around a point $p \in M$

A Co-ordinate Chart around p is a pair (U, φ), where

- U is an open subset of M containing p, and
- $\varphi : U \to \tilde{U}$ is a homeomorphism, where $\tilde{U}$ is an open subset of $\mathbb{R}^d$.



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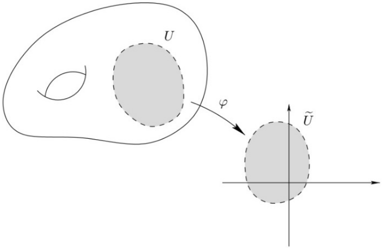

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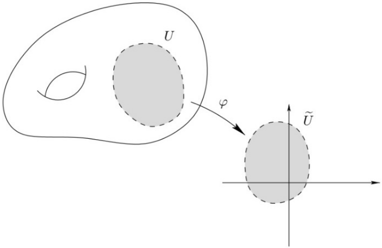
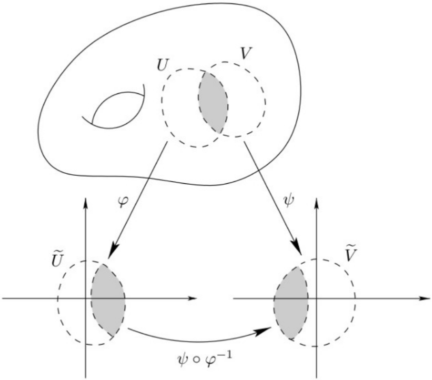


Transition Function

Let (U, φ) & (V, ψ) be two Charts around a point p with $U \cap V \neq \phi$. The functions $\psi \circ \varphi^{-1}$ & $\varphi \circ \psi^{-1}$ are called Transition functions.



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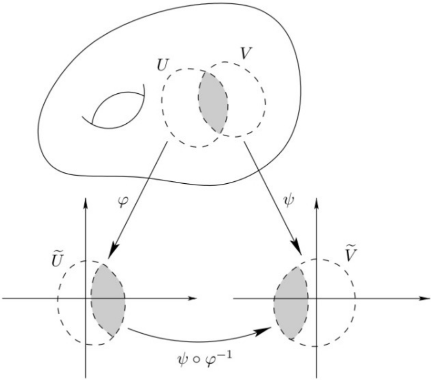

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- $M = \bigcup_{i \in \Lambda} U_i$
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Any Topological d-Manifold M together with a Differential structure on it, is said to be a Smooth d-Manifold.

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Smooth Map between two Smooth Manifolds

Let $(M, \mathcal{U})$ & $(N, \mathcal{V})$ be two Smooth Manifolds of dimension k and l. A function $f : M \to N$ is smooth at $p \in M$ if

- there exists a chart $(U, \varphi) \in \mathcal{U}$ around p &
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s.t. the following map is a smooth map

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

Remark: This definition is independent of choice of charts containing p and $f(p)$.

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Diffeomorphic Maps & Diffeomorphism:

A smooth map $f : M \to N$ between Manifolds of same dimension is said to be a diffeomorphism if

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If $f : M \to N$ is a diffeomorphism then we say M **to diffeomorphic to** N, written as $M \approx N$.

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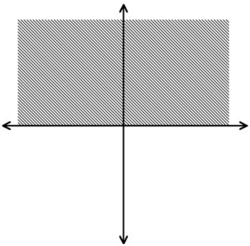

Prototype: The closed Upper-half Space, $\mathbb{H}^d$

$\mathbb{H}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d \geq 0\}$; $\partial\mathbb{H}^d = \{(x_1, \ldots, 0) \in \mathbb{R}^d\}$

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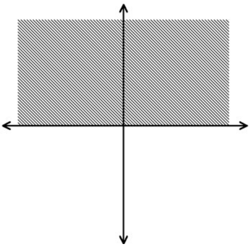

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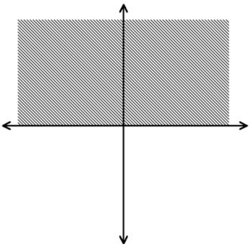

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Defining Smooth Manifolds with Boundary

We take M to be a 2^{nd} countable, T_2 topological space s.t

- For any $p \in M$, $\exists$ an open subset $U \subset M$ around p homeomorphic to an open subset of $\mathbb{H}^d$
- The Atlases on M are modeled on $\mathbb{H}^d$, instead of $\mathbb{R}^d$.

Question: When do we say that $p \in M$ belongs to the boundary, ∂M of M?

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The Boundary of a d-Manifold

$p \in M$ belongs to the boundary ∂M if we have a chart (U, φ) around p, with $\varphi(p) \in \partial \mathbb{H}^d$.

The well-definedness of this definition comes from the Invariance of Domain Theorem.

Boundary of Manifold is again a Manifold

If M is a d-manifold with boundary, then ∂M is a $(d-1)$-manifold without boundary.

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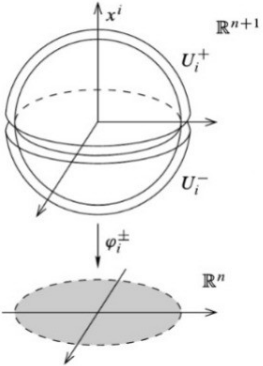
Example 1 : The unit Sphere S^n

The unit sphere $S^n \subset \mathbb{R}^{n+1}$, $n \geq 1$ is a smooth manifold of dimension n.



An Atlas $\mathcal{U}$ on S^n is given as follows:

$$\mathcal{U} = \{(U_i^+, \varphi_i^+), (U_i^-, \varphi_i^-) \ : \ 1 \leq i \leq n+1\}$$

$$U_i^+ = \{(x_1, \ldots, x_i, \ldots, x_{n+1}) \in S^n \mid x_i > 0\},$$

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$$\varphi_i^{\pm}(x_1, \ldots, x_i, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}).$$

Note:

$S^0 = \{-1, 1\}$ is a 0-dimensional Manifold.

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Example 2: The Real Projective Space

- The n-dimensional real projective space $\mathbb{R}P^n$, is defined as the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$, where the equivalence relation is defined as follows:
- $(a_1, \ldots, a_{n+1}) \sim (b_1, \ldots, b_{n+1})$ if $\exists$ a real number $\lambda (\neq 0)$ such that $b_i = \lambda a_i$
- Equivalence class of a point $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$ is denoted by $[x_1, \ldots, x_{n+1}] \in \mathbb{R}P^n$, called homogeneous co-ordinates.

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Example 3 : Complex Projective Space

The n-dimensional complex projective space $\mathbb{C}P^n$, is defined as the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$, where the equivalence relation is defined in a similar way:

$$(a_1, \ldots, a_{n+1}) \sim (b_1, \ldots, b_{n+1}) \text{ if } \exists \text{ a complex number } \lambda (\neq 0) \text{ such that } b_i = \lambda a_i$$

Replacing $\mathbb{R}$ with $\mathbb{C}$ in the atlas (defined above) of $\mathbb{R}P^n$ we obtain an Atlas for $\mathbb{C}P^n$.

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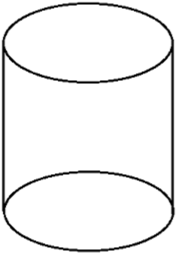

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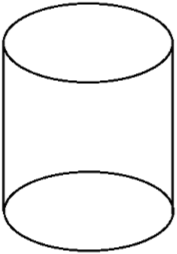

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Example 5: The Dold Manifold $P(m, n)$

Let $\mathbb{C}P^n$ be the complex projective space with homogeneous coordinates $[z_1, \ldots, z_{n+1}]$ as mentioned in Example 2.

Consider the manifold $S^m \times \mathbb{C}P^n$.

The group $\mathbb{Z}_2$ acts on this product space by

$$(x, [z_1, \ldots, z_{n+1}]) \mapsto (-x, [\bar{z}_1, \ldots, \bar{z}_{n+1}]).$$

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Goal

Classify **Compact, Smooth, without Boundary d-dimensional Manifolds** upto a relation called Cobordism.

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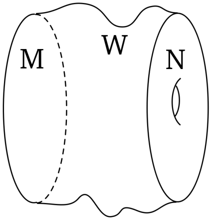
Defining the Cobordism relation

Two Smooth, Closed d-Manifolds M and N are said to be cobordant, if $M \coprod N$ is a boundary of some compact $(d+1)$-Manifold W, i.e., $\partial W \approx M \coprod N$



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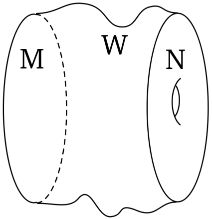

- $\sim$ is an equivalence relation on the set of Smooth, Closed d-dimensional Manifolds.
- Denote the cobordism class of a manifold M by $[M]$.
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- For each d ($d \geq 0$), $\mathcal{N}_d$ **is an abelian group** under the addition:

$$[M] + [N] = [M \coprod N].$$

- **Additive Identity:** Cobordism class of S^d, i.e., $[S^d]$

- For all $[M] \in \mathcal{N}_d$ is of order 2, i.e., $[M] + [M] = [S^d]$

  Set $W = D^{d+1} \coprod M \times [0,1]$; $\partial W = S^d \coprod (M \coprod M)$

- As a result, $\mathcal{N}_d$ becomes a vector space over $\mathbb{Z}_2$.

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The Unoriented Cobordism Algebra

Consider the following vector space over $\mathbb{Z}_2$:

$$\mathcal{N}_* = \oplus_{d \geq 0} \mathcal{N}_d.$$

We define a graded multiplication on $\mathcal{N}_*$ as follows

$$\mathcal{N}_k \times \mathcal{N}_l \to \mathcal{N}_{k+l} \quad \text{by} \quad ([M],[N]) \mapsto [M \times N].$$

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Determining the structure of $\mathcal{N}_*$

René Thom in 1954 completely determined the structure of this Algebra. He proved that

$\mathcal{N}_*$ is a polynomial algebra over $\mathbb{Z}_2$

$$\mathcal{N}_* = \mathbb{Z}_2[x_i \mid 1 \leq i;\ \forall j \in \mathbb{N},\ i \neq 2^j - 1]$$

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- Consider the class of Closed Smooth d-Manifolds equipped with a smooth action of a group G with finite number of fixed points.

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- Notation: (M, η) and (N, ζ) denotes two G-manifolds of dimension d, where $\eta : G \times M \to M$ and $\zeta : G \times N \to N$ are the action maps.

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Note: The action ε need not have finite fixed point set.

This relation $\sim$ gives an equivalence relation on the set of all *Closed, Smooth d-dimensional G-Manifolds.*

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$$Z_d(G) := \{\textit{Closed, Smooth d-dimensional G-Manifolds}\} \Big/ \sim$$

class of (M, η) is denoted by $[M, \eta]$.

The aim is to define a $\mathbb{Z}_2$-graded algebra over

$$Z_*(G) := \bigoplus_{d \geq 0} Z_d(G)$$

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$$\mathrm{Z}_*(\mathrm{G}) := \oplus_{\mathrm{d} \geq 0} \mathrm{Z}_\mathrm{d}(\mathrm{G})$$

The $\mathbb{Z}_2$-graded algebra on $Z_*(G)$ is defined as follows

- **Addition:** $[M,\eta] + [N,\zeta] := [\, M \coprod N \,,\ \eta \coprod \zeta \,]$
- **Multiplication:** $[M,\eta] \times [N,\zeta] := [\, M \times N \,,\ \eta \times \zeta \,]$

The Main Goal:

Determine the structure of $Z_*(G)$ for $n \geq 2$; *where* $G = (\mathbb{Z}_2)^n$

This has been a long standing open-problem.

The $\mathbb{Z}_2$-graded algebra on $Z_*(G)$ is defined as follows

- **Addition:** $[M,\eta] + [N,\zeta] := [\, M \coprod N \,,\ \eta \coprod \zeta \,]$
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For n=2, i.e., when $G = (\mathbb{Z}_2)^2$

Conner and Floyed in [1] proved that for $n = 2$, $Z_*(G)$ **is isomorphic to the Polynomial Algebra with one generator in degree 2**, i.e.,

$$Z_*(G) \cong \mathbb{Z}_2[x]$$

The algebra generator x corresponds to $[\mathbb{R}P^n, \eta]$, where the action η of $G(= \mathbb{Z}_2 \times \mathbb{Z}_2)$ on $\mathbb{R}P^n$ is stated in the following slide.

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Current State of the Art for $n \geq 3$

Understanding the structure of $Z_*(G)$ for $n \geq 3$ (How?)

One way is to map this Algebra to some known Algebra. In this direction one has the 'forgetful' homomorphism:

$$\epsilon_* : Z_*(G) \to \mathcal{N}_* \quad \text{defined as} \quad [M, \eta] \mapsto [M]$$

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- Obtained a sufficient criterion for an element of $Z_*(G)$ to be indecomposable.
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- They also proved a sufficient criterion for a subset $A \subset Z_*(G)$ to be algebrically independent.

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Thank You