I Pigeon-hole Principle (PHP)

If m pigeons are placed in n pigeon-hole, then there is a pigeon-hole that contain at least $\frac{m}{n}$ pigeons

More generally, if $f: A \to B$, then $\exists$ an element $b \in B$ s.t.

$$|f^{-1}(b)| \geq \frac{|A|}{|B|}$$

Pf If not, then for every $b \in B$

$$|f^{-1}(b)| < \frac{|A|}{|B|}$$

$$\sum_{b \in B} |f^{-1}(b)| < \frac{|A|}{|B|} \times |B| = |A|$$

$$|A| < |A| \quad \text{Contradiction.}$$

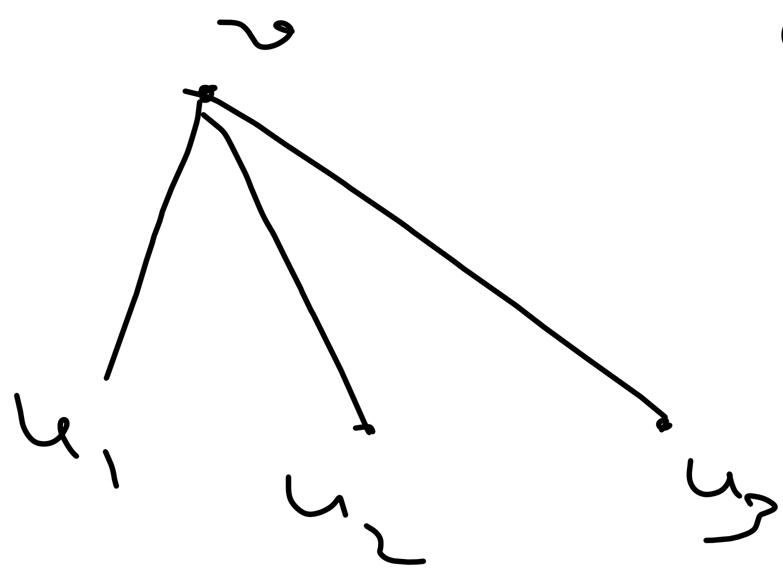
Ex 1 In a graph G with 6 vertices, either there is a triangle or there is an independent set with at least 3 vertices

Pf Fix a vertex v in G.

The remaining 5 vertices are either neighbours of v or non-neighbours of v. Denote these by A & B

By PHP, one of A or B contains at least 3 vertices.

Case 1. Suppose A contains ≥ 3 vertices



If for some u_i, u_j there is an edge $\{u_i, u_j\}$ then we obtain a Δ.

O.W. $\{u_1, u_2, u_3\}$ form an independent set

Ex2 In a graph with n vertices $\exists$ a pair of vertices u, v s.t.

$\deg(u) = \deg(v)$

pf. Let a fn f on V as follows,

$$f(v) = \deg(v)$$

Either $f : V \longrightarrow \{0, 1, 2, \ldots, n-2\}$

or $f : V \longrightarrow \{1, 2, \ldots, n-1\}$

By PHP, $\exists\, u \neq v$ s.t.

$$f(u) = f(v).$$

Ex 3 If $n+1$ elements are chosen from the set $A = \{1, 2, \ldots, 2n\}$.

Then $\exists\, a \neq b$ in A s.t. $a \mid b$ or $b \mid a$.

pf. Let $x_1, x_2, \ldots, x_{n+1}$ be the nos. selected.

For each x_i, write x_i as

$$x_i = 2^s \cdot t, \text{ where } t \text{ is odd}.$$

Define a map $f : \{x_1, \ldots, x_{n+1}\} \longrightarrow$ odd integers in A

by $f(x_i) = t$, the unique odd part of x_i.

By PHP, $\exists\ x_i \neq x_j$ s.t.

$$f(x_i) = f(x_j)$$

$$x_i = 2^{s} t$$

$$x_j = 2^{s'} \cdot t$$

Ex 4. Given a graph G, let

$\alpha(G) =$ independence no.

$\chi(G) =$ chromatic no.

If G has n vertices, then

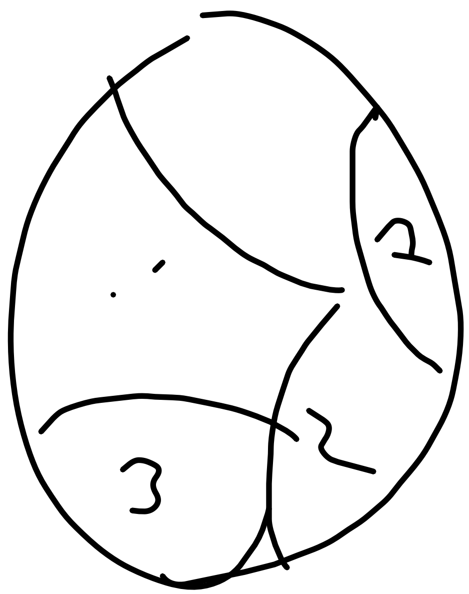
$$n \leq \alpha(G) \cdot \chi(G)$$

In particular, if G is planar then

$$\alpha(G) \geq \frac{n}{4}$$

Pf Colour the vertices of G with $\chi(G)$ colours. Consider the colour classes

V



Clearly, there are $\chi(G)$ colour classes.

The card. of each colour class $\leq \alpha(G)$

Summing up we have

$$n \leq \chi(G) \cdot \alpha(G).$$

Ex 5 For every real $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$

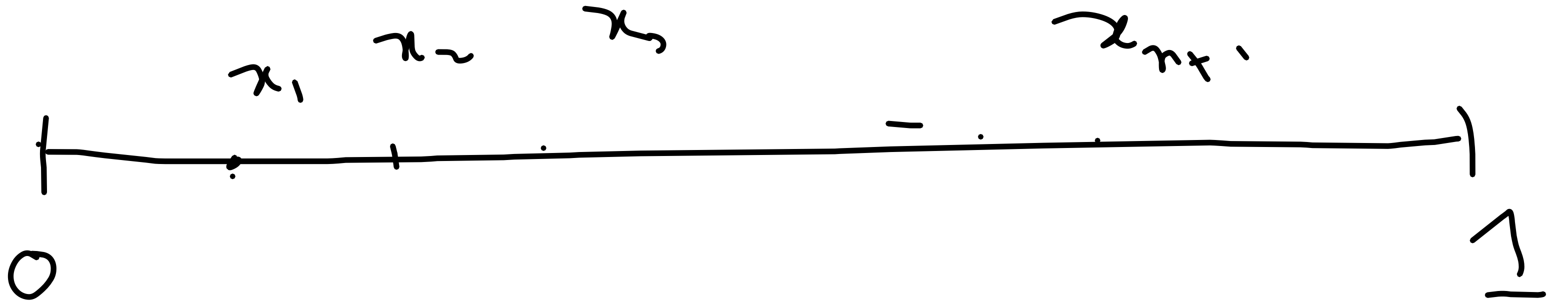
$\exists$ $p, q \in \mathbb{N}$, $1 \le q \le n$ s.t.

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qn} \le \frac{1}{q^2}$$

Pf Fix an irrational α and $n \in \mathbb{N}$.

For $i = 1, 2, \ldots, n+1$, let

$$x_i = i\alpha - \lfloor i\alpha \rfloor.$$ Clearly $0 < x_i < 1$.



By PHP $\exists x_i \neq x_j$, x_i, x_j are in the same subinterval & hence

$$|x_j - x_i| < \frac{1}{n}.$$

$$\left| \left(j\alpha - \lfloor j\alpha \rfloor \right) - \left(i\alpha - \lfloor i\alpha \rfloor \right) \right| < \frac{1}{n}.$$

or $$\left| (j-i)\alpha - \left(\lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor \right) \right| < \frac{1}{n}.$$

Set $q = j - i$

$p = \lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor$

Clearly, $1 \leq q \leq n$. &

$$|q\alpha - p| < \frac{1}{n} \Rightarrow \left|\alpha - \frac{p}{q}\right| < \frac{1}{qn}.$$

Ex 6 (Erdős —)

Given any sequence of $mn+1$ reals, say $a_1, a_2, \ldots, a_{mn+1}$, there is a monotonically increasing seq^n of length m or a monotonically decreasing seq^n of length n.

pf $A = \{a_1, a_2, \ldots, a_{mn+1}\}$.

Define a fn f on A as follows:

$$f(a_i) = (s_i, t_i),$$ where s_i is

the max. length of an increasing seq^n starting from a_i & t_i is the max. length of a decreasing seq^n starting from a_i

If for some i, $s_i > m$ or $t_i > n$ then we are done.

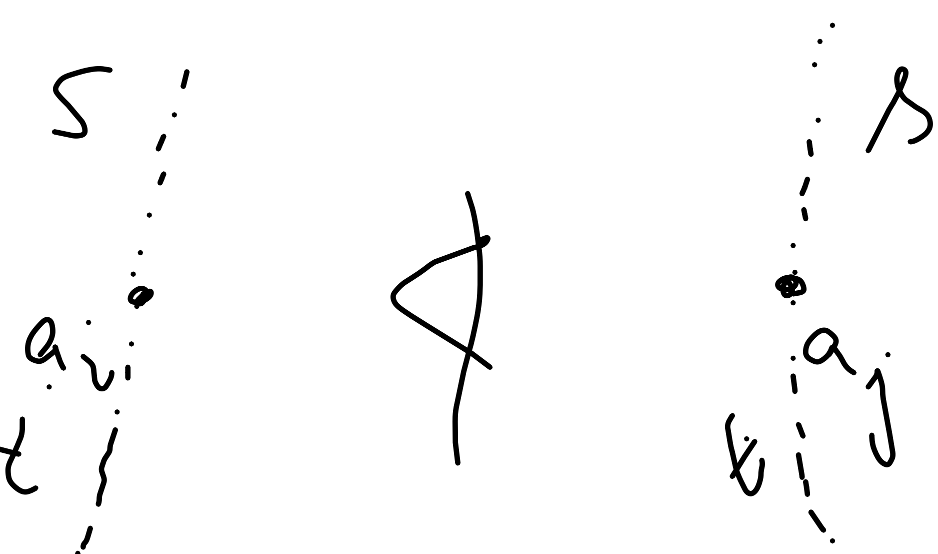
So assume $s_i \le m$ and $t_i \le n$ $\forall i$

So $f : A \longrightarrow \{1, 2, \dots, m\} \times \{1, \dots, n\}$

By PHP, $\exists$ $a_i \neq a_j$ s.t.

$$f(a_i) = f(a_j) = (s, t), \text{ say.}$$

If $a_i > a_j$ we obtain a decreasing seq$^{\underline{n}}$ of length $t+1$ starting from a_i



If $a_i < a_j$ we obtain an increasing seq$^{\underline{n}}$ of length $s+1$ starting from a_i

Ex. Given any 5 lattice points $P_1, \ldots, P_5$

$\exists$ a pair P_i, P_j whose mid-point

is also a lattice point.

$P = (x_1, y_1)$

$Q = (x_2, y_2)$

Mid-pt. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

is a lattice pt.

iff x-coordinates

& y-coordinates

both have same

parity.

Pf Let $P_i = (x_i, y_i)$ $i = 1, \ldots 5.$

$$f(P_i) = (x_i \bmod 2, y_i \bmod 2)$$

$$f : \{P_1, P_2, \ldots, P_5\} = \{(0,0), (0,1), (1,0), (1,1)\}$$

PHP, $\exists\, i \neq j$ s.t

$$f(P_i) = f(P_j)$$

Ex 1 Generalize the PHP to relations.

Ex 2 Suppose a graph G with n vertices has n^2+1 edges. Show that G must contain a triangle

Principle of Inclusion and Exclusion (PIE)

Given a seq^{n} of sets $A_1, A_2, \ldots, A_n$.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \le k_1 < k_2 \cdots < k_j \le n} \left|A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_j}\right|,$$

Where the summation is taken over all seq^{s} of length j in $\{1, \ldots, n\}$.

Pf Fix an element x.

If $x \notin \bigcup A_i$, x is not counted in both LHS & the RHS.

So assume $x \in \bigcup_{i=1}^{n} A_i$

Clearly x is counted exactly once in the LHS expr.

Consider the set $J = \{ j : x \in A_j \}$.

$|J| = m$.

Let $J = \{ g(1), \ldots, g(m) \}$

If in the RHS exp^n one of $k_i \notin \overline{J}$, then x is not counted.

So Assume that each $k_i \in \overline{J}$

So Clearly, there are $\binom{m}{j}$ terms on the RHS.

So x is counted $\binom{m}{j}$ times in the $2^{\underline{nd}}$ summation.

Hence the total count for x is

$$\sum_{j=1}^{m} (-1)^{j+1} \binom{m}{j}$$

$$= \binom{m}{1} - \binom{m}{2} + \binom{m}{3} + \cdots + (-1)^{m+1}\binom{m}{m}$$

$$= -\left\{1 - \binom{m}{1} + \binom{m}{2} + \cdots + (-1)^{m}\binom{m}{m}\right\} + 1$$

$$= -(1-1)^{m} + 1 = 1$$