First-order Logic

The language of first-order logic is a triplet $\mathcal{L} = \langle \mathcal{R}, \mathcal{F}, \mathcal{C} \rangle$, where

$\mathcal{R} = \{ r_1, r_2, r_3, \dots \}$ is a ctble set of relation symbols

$\mathcal{F} = \{ f_1, f_2, \dots \}$ is a ctble set of fn symbols

$\mathcal{C} = \{ c_1, c_2, \dots \}$ a ctble set of constants

With each $r \in R$ and $f \in \mathcal{F}$ we associate an integer, called the <u>arity</u>.

We also assume that "$=$" is also part of $\mathcal{L}$. We also have a ctble set of variables $\mathcal{V} = \{x_1, x_2, \ldots\}$.

<u>Def^n</u> The set of terms is the smallest set satisfying the following

- Each constant $c \in \mathcal{C}$ is a term
- Each variable $x \in \mathcal{V}$ is a term.
- If f is an n-ary fn symbol & $t_1, t_2, \ldots t_n$ are terms, then

$$f(t_1, t_2, \ldots, t_n) \text{ is a term.}$$

Def^n The atomic formulas of $\mathcal{L}$ are defined as follows.

- If t_1 and t_2 are terms, then $\boxed{t_1 = t_2}$ is an atomic formula
- If p is an n-ary relation symbol & $t_1, t_2, \ldots, t_n$ are terms, then $\boxed{p(t_1, \ldots, t_n)}$ is an atomic formula.

Def^n The set of formulas of the first-order logic is the smallest set $\Phi_{\mathcal{L}}$ satisfying the following.

- All atomic formulas $\varphi \in \Phi_{\mathcal{L}}$.
- If $\varphi \in \Phi_{\mathcal{L}}$ then $\neg\varphi \in \Phi_{\mathcal{L}}$.
- If φ and ψ are in $\Phi_{\mathcal{L}}$ then $\varphi \vee \psi \in \Phi_{\mathcal{L}}$.
- If $\varphi \in \Phi_{\mathcal{L}}$ and $x \in \mathcal{V}$, then $(\exists x \varphi) \in \Phi_{\mathcal{L}}$.

We also define

$$\varphi \wedge \psi \overset{\text{def}^n}{=} \neg(\neg\varphi \vee \neg\psi)$$

$$\varphi \rightarrow \psi \overset{\text{def}^n}{=} (\neg\varphi \vee \psi).$$

$$\varphi \leftrightarrow \psi \overset{\text{def}^n}{=} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

$$\forall x\, \varphi \overset{\text{def}^n}{=} \neg \exists x (\neg\varphi).$$

$\forall \rightarrow$ universal quantifier.

$\exists \rightarrow$ existential quantifier.

First-order structure of $\mathcal{L}$ is a pair

$\mathcal{M} = \langle S, i \rangle$, where S is a non-empty set and i is a fn on $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ satisfying the following

- For any n-ary relation symbol $r \in \mathcal{R}$, $i(r)$ is an n-ary relation on S
- For any n-ary fn symbol $f \in \mathcal{F}$, $i(f)$ is an n-ary fn on S. $i(f): S^n \to S$

- If $c \in \mathcal{C}$, then $i(c) \in S$.

$$i(r) := r^{\mathcal{M}}.$$

$$i(f) := f^{\mathcal{M}}.$$

$$i(c) := c^{\mathcal{M}}$$

Interpretation

An interpretation $\mathcal{I}$ is a pair $\langle \mathcal{M}, \sigma \rangle$, where $\mathcal{M}$ is a first-order structure & σ is a fn $\mathcal{V} \longrightarrow S$.

Defn Let $\mathcal{I}$ be an interpretation as above & let t be a term. We define $t^{\mathcal{I}}$.

- If t is a const, $t^{\mathcal{I}}$ is $t^{\mathcal{M}}$.
- If t is a variable, then $t^{\mathcal{I}}$ is $\sigma(t)$

· If t is of the form $f(t_1, \dots, t_n)$

Then $t^{\mathcal{I}}$ is $f^{\mathcal{M}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}})$.

Interpretation of formulas in $\Phi_{\mathcal{L}}$.

Let ϕ be a formula of first-order theory & $\mathcal{I} = \langle \mathcal{M}, \sigma \rangle$ be an interpretation. The notion of a formula ϕ being satisfied by $\mathcal{I}$ written as $\mathcal{I} \models \phi$ is defined recursively as follows.

- If ϕ is an atomic formula of the form $p(t_1, t_2, \ldots, t_n)$, then $\mathcal{I} \models p(t_1, \ldots, t_n)$ if $p^{\mathcal{I}}(t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}})$
- If φ is $\neg \psi$, $\mathcal{I} \models \varphi$ if $\mathcal{I} \not\models \psi$
- If φ is $\psi_1 \vee \psi_2$, we define $\mathcal{I} \models \varphi$ if $\mathcal{I} \models \psi_1$ or $\mathcal{I} \models \psi_2$
-

- If φ is $\exists x \psi$, then we say $\mathfrak{I} \models \varphi$ if $\exists x^* \psi$ then we say $\mathfrak{I} \models \varphi$ if for some $s \in S$

$$\mathfrak{I}[x^* \to s] \models \psi$$

$\mathfrak{I} = \langle M, \sigma \rangle$, where $\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \neq x^* \\ s & \text{if } x = x^* \end{cases}$

Def^n φ is satisfiable, if there is an interpretation $\mathcal{I}$ s.t. $\mathcal{I} \models \varphi$

φ is said to be valid if for all interpretation $\mathcal{I}$, $\mathcal{I} \models \varphi$

If $\mathcal{I} \models \varphi$, then $\mathcal{I}$ is called a model for φ

<u>Defⁿ</u> (Free and bound variables)

Given a formula of the form $\exists x\, \varphi$,
the formula φ is said to be under
the scope of x.

A variable x is said to be free
if it is <u>not</u> in the scope of $\exists x$
o.w. x is said to be bound.

If φ has no free variables, then
φ is a sentence.

We write $\varphi(x_1, \ldots, x_n)$ if the free variables of φ are among $x_1, \ldots, x_n$.

Axiomatisation of 1st order logic.

A1 All tautologies of Prop. Logic.

A2 (a) $x = x$

(b) $(t_1 = t_2) \longrightarrow (\phi(t_1) = \phi(t_2))$, where ϕ is atomic.

A3 $\phi(t) \longrightarrow \exists x\, \phi$

Rules of Inference

I M.P.

$$\frac{\alpha,\ \alpha \to \beta}{\beta}.$$

II Generalisation:

$$\frac{\varphi(x) \longrightarrow \psi}{\exists x\, \varphi \longrightarrow \psi},$$

provided x is <u>not</u> free in ψ

Thm (Gödel)

$\models \varphi$ iff $\vdash \varphi$.