Moore graphs - $g = 2d+1$.

↳ Regular.

Petersen graph $\longrightarrow$ $srg(10, 3, 0, 1)$.

$srg(v, k, \underline{\lambda}, \underline{\mu})$.

any two adjacent vertices have λ common neighbours.

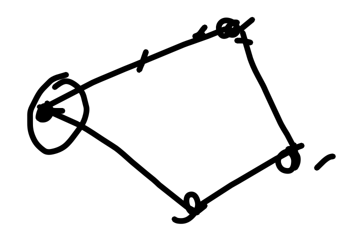
'' '' non-adjacent '' '' μ ''

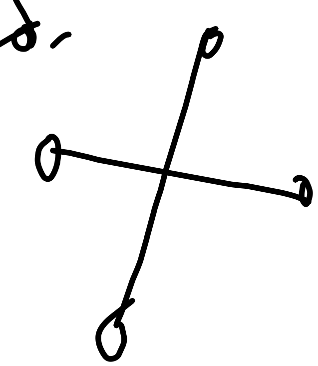
''

Exc No of vertices of a Moore graph: $1 + k \sum_{i=0}^{d-1} (k-1)^i$

Hint (use BFS)

$G = srg(v, k, \lambda, \mu)$





$\bar{G} = srg(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$

(Exc)

1° $srg(v, k, \lambda, \mu)$.

$$(v-k-1)\mu = k(k-\lambda-1).$$

Level 0 $\longrightarrow$ root

level 1 $\longrightarrow$ k.

$$k(k-\lambda-1) = (v-k-1)\mu.$$

level 2 $\longrightarrow$ $(v-k-1)$

$srg(v, k, \lambda, \mu)$

Adjacency matrix A, I, J

Graph is regular $\longrightarrow$ (1) $\underline{AJ = JA = kJ}$

$$AJ = \begin{pmatrix} - & - & - & - \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

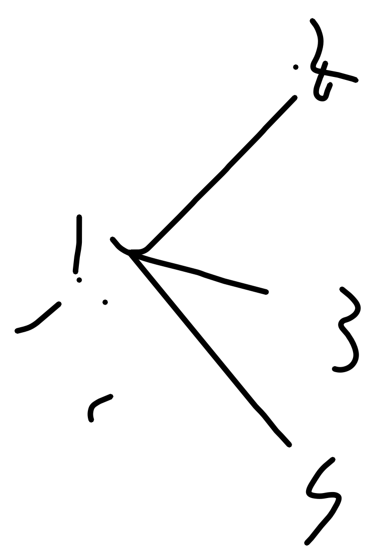
$(1,1) + (1,2) + \cdots + (1,v)$

$= 1 + \cdots + 1 = k$

② $A^2 = \kappa I + \lambda A + \mu(J - I - A)$.

(i, j)th element gives the no of 2-step walks from i to j

$(1,1) \to \kappa$. no of 2-step walks from $i \to i$



(i, j) the no of 2-step walks from i to j if they are connected by an edge.

no of 2.S. walks from i to j if they are not connected

mutually exclusive and exhaustive

A is real symmetric.

→ real eigenvalues

→ orthogonal eigenvectors.

$AJ = JA = kJ$.

k - e v

eigen vector ← $\underline{x}$, $Jx = 0$

$\sum x_i = 0$

$$A^2 x = k \underline{I} x + \lambda A x + \mu (J - \underline{I} - A) x.$$

$$Jx = 0. \qquad Ax = p x.$$

$$\Rightarrow p^2 x = k x + \lambda p x - \mu x - \mu p x$$

$$\Rightarrow p^2 + (\mu - \lambda) p - (k - \mu) = 0.$$

$$p = \frac{1}{2}\left[(\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right].$$

$\underline{k}$, $\underline{r}$, $\underline{s}$

trace $A = 0$ = Sum of eigenvalues.

$$k + fr + gs = 0 \quad \text{--- } 1$$

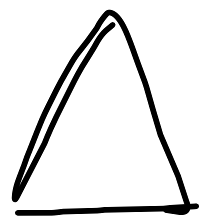
$$1 + f + g = v \quad \text{--- } 2$$

$$f = \frac{1}{2}\left[(v-1) - \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right]$$

$$g = \qquad +$$

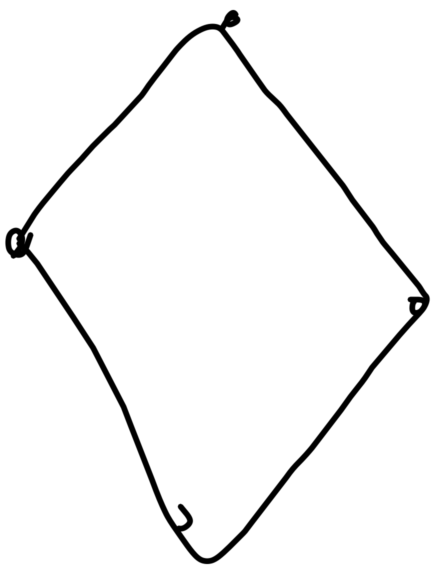
Strongly regular Moore graphs of girth $= 5$.

$\mathrm{srg}(v, k, \overset{0}{\overset{\|}{\lambda}}, \underset{\underset{1}{\|}}{\mu})$



$\mathrm{srg}(v, k, 0, 1)$

$k(k - \lambda - 1) = \mu(v - k - 1)$



Substitute $\lambda = 0$, $\mu = 1$

$v = k^2 + 1$

$$\mu_{\pm} = \frac{1}{2}\left[k^2 \pm \frac{2k-k^2}{\sqrt{4k-3}} \right]$$

must be a rational no.

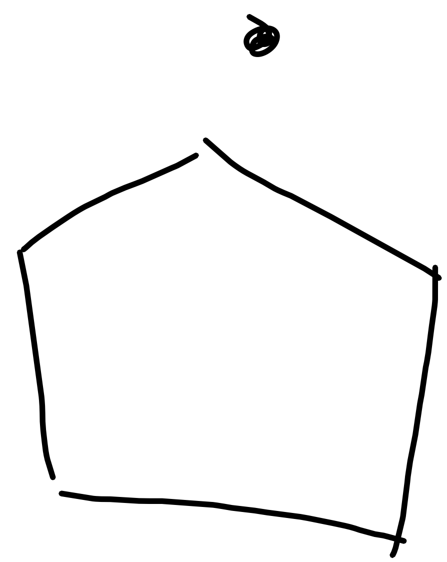
1°

$$2k - k^2 = 0$$

$$k = 0, 2.$$

$$k = 0, \quad v = 1$$

$$k = 2, \quad v = 5$$



2° $\sqrt{4k-3} = t \Rightarrow k = \frac{t^2+3}{4}$

Substituting the value of k;

$$M_\pm = \frac{1}{2}\left[\left(\frac{t^2+3}{4}\right)^2 \pm \frac{\frac{t^2+3}{4} - \left(\frac{t^2+3}{4}\right)^2}{t}\right]$$

$$\Rightarrow 32 M_\pm = t^4 + 6t^2 + 9 \pm \left(-t^3 + 2t + \frac{15}{t}\right)$$

$$t \mid 15$$

$t \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$ $k \in \{1, 3, 7, 57\}$

$k=1, v=2$ 

$k=3, v=10 \longrightarrow$ Petersen graph.

$k=7, v=50 \longrightarrow$ Hoffman-Singleton graph.

$k=57, v=3250 \longrightarrow$ (open)

<u>Theorem</u> (Hoffman-Singleton)

there are no strongly regular girth 5 Moore graphs other than the ones <u>described</u>

Adjacency matrix

$$A \longrightarrow \underbrace{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n}_{\text{Spectrum of the graph.}}$$

$\lambda_1 - \lambda_2 \longrightarrow$ Spectral gap.

$\lambda(G) = \max \{ |\lambda_2|, |\lambda_n| \}$ Spectral radius.

graphs which satisfy $\underline{\lambda(G) \leq 2\sqrt{d-1}}$

Ramanujan graphs