Ex 1. There are two drawers in each of the 3 boxes that are identical in appearance.
The 1st box contains a gold coins in each of the drawers. The 2nd one contains a silver coin in each drawer & the 3rd contain a gold coin in one drawer & a silver coin in the other. A box is chosen, one of its drawer opened & a gold coin is found. What is the prob. that the other drawer contains a gold.

Ex 2. In answering a question in a multiple choice test, an examinee either know the answer with prob. p or he guesses with prob. $q = 1-p$. Let the prob. of answering the question correctly be 1 for an examinee who knows the answer and $\frac{1}{4}$ for one who guesses. Suppose an examinee answer a question correctly. What is the prob. that he really knows the answer?

Ex 3 The flow of traffic at a certain street crossing is s.t. the prob. of a car's passing during a second is p and there is no interaction between the passing of cars at different seconds. Treating seconds as indivisible & supposing a pedestrian is to can cross the street only if no car pass during the next 3 seconds. Find the prob. that the pedestrian has to wait for exactly $k = 0, 1, 2, 3, 4$ seconds.

Def^n. A function whose domain is a sample space & whose range is a finite or ctbly infinite set of real nos is called a discrete random variable.

e.g. The sum of the points on the 2 dice thrown simultaneously is a discrete random variable.

$$X((i,j)) = i+j$$

$$\Omega = \{(i,j).\ 1 \le i,j \le 6\}.$$

range of X $= \{2, 3, \ldots, 12\}$.

Consider n independent Bernoulli trails. Then the no. of successes is an example of a discrete r.v.

Def^n. (Ω, P) a prob. space.

Let X be a discrete r.v. on Ω.

With each x, we associate a prob., denoted by p_X & called a prob. density fn (pdf) & is given by

$$p_X(k) = P(\{s \in \Omega : X(s) = k\}) = \boxed{P(X=k)}$$

Defn (Ω, P). X: discrete r.v.

We define the distribution fn or prob. distribution of the r.v. X, denoted by $F \mid F_X$, is defined as follows:

$$F(x) = \boxed{P(X \le x) = P(\{s \in \Omega : X(s) \le x\})}$$

e.g. X a Binomial r.v. The pdf of X is

$$p_X(k) = P(X = k) = \binom{n}{k} p^k q^{n-k}.$$

Thm (a) $F \uparrow$.

(b) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$.

(c) $F(+\infty) = \lim_{x \to \infty} F(x) = 1$.

(d.) $F(x+0) = F(x)$ i.e. F is right cts.

Pf (a) Let $x \le y$

$A_x = \{s \in \Omega : X(s) \le x\} \subseteq A_y = \{s \in \Omega : X(s) \le y\}$

$P(X \le x) \le P(X \le y)$.

i.e. $F(x) \le F(y)$.

(b) Let $A_n = \{ s \in \Omega : X(s) \le -n \}$.

$A_n \downarrow$ & $\lim A_n = \bigcap_{n=1}^{\infty} A_n = \phi$

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right) = P(\phi) = 0.$$

$$\downarrow$$

$$\lim_{x\to -\infty} F(x) = F(-\infty)$$

(d) Fix x

Let $A_n = \{ s \in \Omega : X(s) \le x + \frac{1}{n} \}$.

$A_n \downarrow$ & $\lim A_n = \bigcap_{n=1}^{\infty} A_n = (X \le x)$

$$\lim F\left(x + \frac{1}{n}\right) = \lim P(A_n)$$
$$= P(\lim A_n) = P(X \le x) = F(x)$$

Defn: Let X be a discrete r.v. The expectation of X, denoted by $E(X)$ / μ / μ_X is defined as follows:

$$\boxed{E(X) = \sum_k k\,P(X=k).}$$

e.g. X e binomial r.v.

$$E(X) = \sum_{k=0}^{n} k \cdot P(X=k)$$

$$= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$

$$= np \cdot \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} q^{n-k}$$

$$= np\,(p+q)^{n-1} = np.$$

Two dice & r.v. Sum of the pts on the dice.

$$E(X) = \sum_{k} k\, P(X=k)$$

$$= 2 \cdot \underline{P(X=2)} + 3\,P(X=3) + \cdots + 12\,P(X=12)$$

$P(X=2) = \frac{1}{36}$ $P(X=5) = \frac{4}{36}$ & etc.

$P(X=3) = \frac{2}{36}$

$P(X=4) = \frac{3}{36}$

$$2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + \cdots + 12 \times \frac{1}{36}$$

$$= 7$$

Thm

(a) If the r.v. X takes const. value c, then $E(X) = c$.

(b) If $X \leq Y$, then $E(X) \leq E(Y)$.

(c) Let X, Y be 2 discrete r.vs. & $a, b \in \mathbb{R}$. Then

$$E(aX + bY) = aE(X) + bE(Y).$$

Pf (c) Enough to show that

$$E(X+Y) = E(X) + E(Y).$$

$$E(aX) = a\,E(X). \quad (\text{Ex})$$

$$E(X+Y) = \sum_{x,y} (x+y)\, P(X=x \text{ and } Y=y).$$

$$= \sum_{x,y} x\, P(X=x; Y=y) + \sum_{x,y} y\, P(X=x; Y=y)$$

$$= \sum_{x} x \sum_{y} P(X=x; Y=y) + \sum_{y} y \sum_{x} P(X=x; Y=y)$$

$$= \sum_x x P(X=x) + \sum_y y P(Y=y)$$

$$= E(X) + E(Y).$$

e.g. Consider n Bernoulli trials where prob. of success in each trial is p.

Let X_i be the r.v. on the ith trial defined as follows:

$$X_i(s) = \begin{cases} 1 & \text{if } s \text{ is a success} \\ 0 & \text{o.w.} \end{cases}$$

$X = \sum x_i$ = counts the no. of successes in n Bernoulli trials.

$$E(X) = \sum E(x_i)$$

$$= \sum_{i=1}^{n} p = np.$$

Ex Let X and Y be independent r.v.s.

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y).$$

$$E(X \cdot Y) = E(X) E(Y)$$

Def^n The median of X is a pt. m s.t.

$$P(X < m) = P(X > m).$$

If $\exists$ two pts. m and m' s.t.

$$P(X \leq m) = P(X \geq m') = \frac{1}{2},$$ the

$\dfrac{m+m'}{2}$ is taken to be the median.

Thm. Let X be a discrete r.v. and g be any fn. Then

$$E(g(X)) = \sum_k g(k) P(X=k).$$

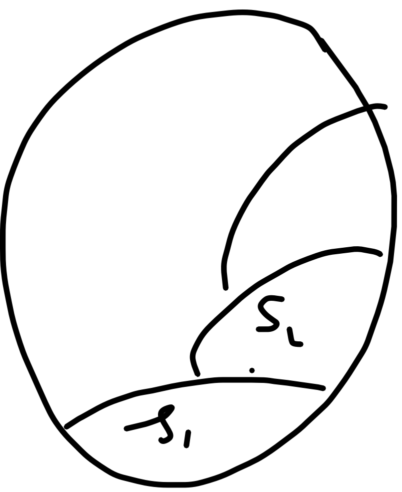
Pf Let $W = g(X)$ & let $w_1, w_2, \ldots$ be the possible values of W.

Let $S_j = \{k : g(k) = w_j\}$.

$$P(W = w_j) = P(X \in S_j)$$

$$\begin{aligned} E(W) &= \sum_j w_j P(W = w_j). \\ &= \sum_j w_j P(X \in S_j). \\ &= \sum_j \sum_{k \in S_j} g(k) P(X = k) \\ &= \sum_k g(k) P(X = k). \end{aligned}$$

Ω.



In particular

$$E(X^2) = \sum_x x^2 P(X=x).$$

<u>Def^n</u> X: discrete r.v.

The variance of X, denoted by $\mathrm{Var}(X)$ is defined by

$$\mathrm{Var}(X) = E\left(X-\mu\right)^2, \text{ where } \mu = E(X).$$

Thm $\mathrm{Var}(X)$ exists iff $E(X)$ and $E(X^2)$ exist and

$$\mathrm{Var}(X) = E(X^2) - \mu^2$$

Pf $\mathrm{Var}(X) = E(X-\mu)^2$

$$= E\left(X^2 - 2\mu X + \mu^2\right)$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2$$

Standard Deviation : $\sqrt{Var(X)}$.

<u>Thm</u> Let $Y = a + bX$. $E(Y) = a + b\mu$.

$$Var(Y) = b^2 Var(X).$$

<u>Pf.</u>
$$Var(Y) = E\left(Y - (a + b\mu)\right)^2$$
$$= E\left(b(X - \mu)\right)^2 = b^2 E(X - \mu)^2$$
$$= b^2 Var(X).$$