Given 2 r.v.s X and Y, the correlation coefficient of X and Y, denoted by $\rho(X,Y) = \mathrm{cov}(X^*, Y^*)$, where

$$X^* = \frac{X-\mu_X}{\sigma_X} \; ; \; Y^* = \frac{Y-\mu_Y}{\sigma_Y}$$

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

<u>Prop.</u> (a) $|\rho| \le 1$.

(b) $|\rho| = 1$ iff $Y = aX + b$, for some consts. a and b.

<u>Pf</u> $0 \le \mathrm{Var}(X^* \pm Y^*) = \mathrm{Var}(X^*) + \mathrm{Var}(Y^*) \pm 2\,\mathrm{Cov}(X^*, Y^*)$

$= \mathrm{Var}(X^*) + \mathrm{Var}(Y^*) \pm 2\rho$.

<u>Claim</u> $\mathrm{Var}(X^*) = \mathrm{Var}(Y^*) = 1$.

$$\mathrm{Var}(X^*) = \mathrm{Var}\left(\frac{X-\mu_X}{\sigma_X}\right)$$

$$= \frac{1}{\sigma_X^2}\mathrm{Var}(X-\mu_X)$$

$$= \frac{1}{\sigma_X^2}\mathrm{Var}(X) = 1$$

Hence $0 \le 2 \pm 2\rho$.

$$\Rightarrow \rho \ge -1 \;\&\; \rho \le 1$$

(b) Suppose $\rho = 1$.

$$\mathrm{Var}(X^* - Y^*)$$

$$= \mathrm{Var}(X^*) + \mathrm{Var}(Y^*) - 2\rho$$

$$= 2 - 2 = 0.$$

$$\mathrm{Var}(X) = \sum_x (x - \mu_X)^2.$$

Hence $X^* - Y^* = c$, for some const. c.

$$\frac{X-\mu_X}{\sigma_X} - \frac{Y-\mu_Y}{\sigma_Y} = c.$$

$\Rightarrow$ X and Y are linearly related.

$\rho = -1$, $Var(X^* + Y^*) = 0$.

Conversely, suppose $Y = aX + b$ for some consts a, b. Then

$$\mu_Y = a\mu_X + b.$$

$$\sigma_Y = |a|\sigma_X.$$

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = \frac{E(aX^2 + bX) - \mu_X(a\mu_X + b)}{|a|\sigma_X^2}$$

$$= \frac{a\left(E(X^2) - \mu_X^2\right)}{|a|\sigma_X^2} = \frac{a\cancel{\sigma_X^2}}{|a|\cancel{\sigma_X^2}} = \pm 1$$

Sample correlation Coeff^n.

Suppose X and Y are r.vs.

& $\rho(X,Y)$ is unknown.

Suppose some values of X, Y

viz $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$

are known. Then we

define the sample correlation coeff^n by

$$r = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{i=1}^{n} y_i}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2} \cdot \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2}}$$

$$= \frac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{\sqrt{n\sum x_i^2 - (\sum x_i)^2} \times \sqrt{n\sum y_i^2 - (\sum y_i)^2}}$$

$$= \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}}$$

Lemma 1. Let f be a real-valued f^n defined on an open interval (a,b). Suppose f is differential at an interior pt $c \in (a,b)$. Then $\exists$ a f^n f^* cts at c s.t.

$$\boxed{f(x) - f(c) = (x-c)\,f^*(x)}$$

with $f^*(c) = f'(c)$

Pf Define f^* on (a,b) by

$$f^*(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{if } x \neq c. \\ f'(c) & \text{if } x = c. \end{cases}$$

$$\lim_{x \to c} f^*(x) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$$

Lemma 2 f is a f^n defined on (a,b)

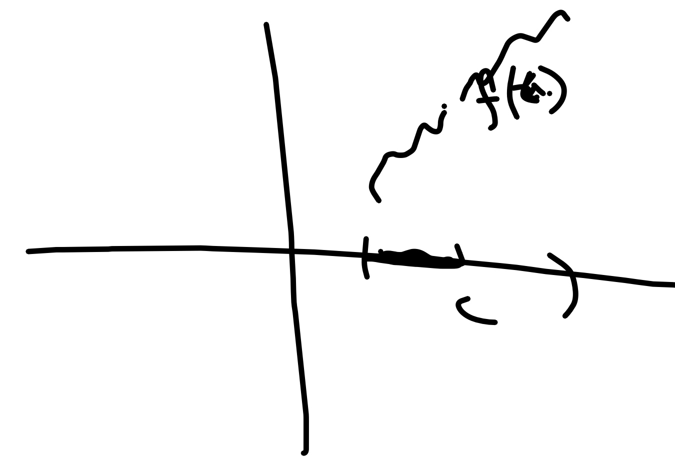
& suppose for some pt. $c \in (a,b)$

$f'(c) > 0$ or $f'(c) = +\infty$.

Then $\exists$ a nbd $N(c) \subseteq (a,b)$

$f(x) > f(c)$ if $x > c$.

& $f(x) < f(c)$ if $x < c$.



Similar result holds when $f'(c) < 0$
or $f'(c) = -\infty$

<u>pf</u>. <u>Case 1</u>. Suppose.

Pf. Case 1. Suppose $f(c) > 0$.

By Lemma 1 $\exists$ a fn f^* cts at c (1) where $f^*(c) = f'(c)$

s.t. $\boxed{f(x) - f(c) = (x-c) f^*(x)}$,

Since f^* is cts at c & $f^*(c) > 0$.

$\exists$ a nbd $N(c)$ of c on which

f^* is +ve. On the nbd $N(c)$,

$f(x) - f(c)$ has the same sign

as $x - c$. $\therefore$ $f(x) > f(c)$ if $x > c$.

$f(x) < f(c)$ if $x < c$.

Case 2. $f'(c) = +\infty$.

Then on a small nbd $N(c)$ of c

we have $$\frac{f(x)-f(c)}{x-c} > 1$$

Hence on $N(c)$ $f(x)-f(c)$ has the same sign as $x-c$

Thm Let f be defined on (a,b) & suppose f has a local maximum or a local minimum at an interior pt. $c \in (a,b)$. If f is diff^{ble} at c,

$$f'(c) = 0$$

Pf. $\times$ (1) $f'(c) > 0$ or $f'(c) = +\infty$.

$\times$ (2) $f'(c) < 0$ or $f'(c) = -\infty$

(3) $f'(c) = 0$

By Lemma 2, cases ① & ② cannot hold. Hence we must have $f'(c) = 0$.

Ex Show that $f'(c) = 0$ is not a sufficient condition for local maximum/min. at c.

Thm Let f be defined on (a,b)

Suppose at some interior pt $c \in (a,b)$

f is n-times differentiable. Also

assume that

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0.$$

$$f^{n}(c) \neq 0.$$

If n is even & $f^{(n)}(c) < 0$, Then f has a local maximum at c.

If n is even & $f^{(n)}(c) > 0$. Then f has a local min. at c.

If n is odd, f neither has a local maximum or a local minimum at c.

If n is even & $f^{(n)}(c) < 0$,
Then f has a local maximum at c.
If n is even & $f^{(n)}(c) > 0$.
Then f has a local min. at c.
If n is odd, f neither has a local maximum or a local minimum at c.

Pf In some nbd $N(c)$ of c, we have by Taylor's expⁿ

$$f(x) - f(c) = \text{terms involving } f^{(i)}(c) \text{ for } i < n$$
$$+ \frac{f^{(n)}(\xi)}{n!}(x-c)^n, \text{ for some } \xi \in N(c)$$

$$\therefore \quad f(x) - f(c) = \frac{f^{(n)}(\xi)}{n!}\underline{(x-c)^n}.$$

If n is even, $(x-c)^n > 0$.

Hence $f(x) - f(c) < 0$ if $f^{(n)}(\xi) < 0$.