**Thm** Suppose f is a real-valued fn defined on an open interval $(a,b)$. Suppose f is n-times continuously differentiable & for some pt $c \in (a,b)$

$$f^{(1)}(c) = f^{(2)}(c) = \cdots = f^{(n-1)}(c) = 0 \ \& \ f^{(n)}(c) \neq 0.$$

If n is even, then f has a local max at c if $f^{(n)}(c) < 0$

& f - - - - min at c if $f^{(n)}(c) > 0$.

If n is odd, then it neither a max or min at c.

# Lagrange's Method of Multipliers.

**Thm** Let $f$ be a real-valued $C^1$-$f^n$ on an open set $S \subseteq \mathbb{R}^n$. Let $g_1, g_2, \ldots, g_m$ be $m$ real-valued $f^n$ s.t. $\underset{\sim}{g} = (g_1, \ldots, g_m) \in C^1$. Assume $m < n$.

Let $X_0 = \{ \underset{\sim}{x} \in S : \underset{\sim}{g}(\underset{\sim}{x}) = 0 \}$

Suppose $\underset{\sim}{x}_0 \in X_0$ & $\exists$ e nbd $N \subseteq S$ of $x_0$ s.t. $\boxed{f(\underset{\sim}{x}) \le f(\underset{\sim}{x}_0)}$ $\forall \underset{\sim}{x} \in N \cap X_0$

or $\boxed{f(\underset{\sim}{x}) \ge f(\underset{\sim}{x}_0)}$, $\forall \underset{\sim}{x} \in N \cap X_0$.

Suppose the $m$-rowed determinant

$$\boxed{\det\left[\frac{\partial}{\partial x_j} g_i(\underset{\sim}{x}_0)\right] \neq 0.}$$

Then $\exists$ consts. $\lambda_1, \lambda_2, \ldots, \lambda_m$ s.t.

$$\boxed{\frac{\partial f(\underset{\sim}{x}_0)}{\partial x_r} + \sum_{k=1}^{m} \lambda_k \cdot \frac{\partial g_k(\underset{\sim}{x}_0)}{\partial x_r} = 0.}$$

for $r = 1, 2, n$

**Thm** $f$ has a local extremum at $\underset{\sim}{x_0}$ subject to the constraint $g(\underset{\sim}{x}) = 0$. Assume that

$$\frac{\partial g(\underset{\sim}{x_0})}{\partial x_1} \neq 0.$$

Then $\exists$ a constant $\lambda$ s.t.

(1) $$\boxed{\frac{\partial f(\underset{\sim}{x_0})}{\partial x_i} - \lambda \frac{\partial g(\underset{\sim}{x_0})}{\partial x_i} = 0}, \text{ for } i = 1, 2, \ldots n.$$

Pf $\underset{\sim}{x} = (x_1, \ldots, x_n)$

$\underset{\sim}{x}_0 = (x_{10}, x_{20}, \ldots, x_{n0})$

$\underset{\sim}{u} = (x_2, x_3, \ldots, x_n)$

$\underset{\sim}{u}_0 = (x_{20}, \ldots, x_{n0})$.

Since $\dfrac{\partial g(\underset{\sim}{x}_0)}{\partial x_1} \neq 0$, by the Implicit Function Theorem, $\exists$ a nbd $N$ of $\underset{\sim}{u}_0$ & a unique cts diff $f^n$ $h$ s.t

- $(h(\underset{\sim}{u}), \underset{\sim}{u}) \in S \quad \forall \underset{\sim}{u} \in N$
- $\boxed{g(h(\underset{\sim}{u}), \underset{\sim}{u}) = 0 \quad \forall \underset{\sim}{u} \in N}$
- $h(\underset{\sim}{u}_0) = x_{10}$

(2) $g(h(\underset{\sim}{u}), \underset{\sim}{u}) = 0$, $\forall u \in N$.

Define $\lambda = \frac{\partial f(\underset{\sim}{x}_0)}{\partial x_1} \Big/ \frac{\partial g(\underset{\sim}{x}_0)}{\partial x_1}$

Hence $\lambda$ satisfies the 1st of $n$ eqns in (1)

Fix $i > 1$. Then by (2),

(3) $$\frac{\partial g(h(\underset{\sim}{u}), \underset{\sim}{u})}{\partial x_i} + \frac{\partial g(h(\underset{\sim}{u}), \underset{\sim}{u})}{\partial x_1} \cdot \frac{\partial h(\underset{\sim}{u})}{\partial x_i} = 0$$

Since $(h(\underset{\sim}{u}_0), \underset{\sim}{u}_0) = \underset{\sim}{x}_0$, by (3) we set

(4) $$\boxed{\frac{\partial g(\underset{\sim}{x}_0)}{\partial x_i} + \frac{\partial g(\underset{\sim}{x}_0)}{\partial x_1} \frac{\partial h(\underset{\sim}{u}_0)}{\partial x_i} = 0.}$$

(5) $$\frac{\partial f(\underset{\sim}{x})}{\partial x_i} = \frac{\partial f(h(\underset{\sim}{u}), \underset{\sim}{u})}{\partial x_i} + \frac{\partial f(h(\underset{\sim}{u}), \underset{\sim}{u})}{\partial x_1} \frac{\partial h(\underset{\sim}{u})}{\partial x_i}$$

$\forall \underset{\sim}{u} \in N$

Since $(h(\underset{\sim}{u}_0), \underset{\sim}{u}_0) = \underset{\sim}{x}_0$, by (3) we set

(4) $$\boxed{\frac{\partial g(\underset{\sim}{x}_0)}{\partial x_i} + \frac{\partial g(\underset{\sim}{x}_0)}{\partial x_1}\frac{\partial h(\underset{\sim}{u}_0)}{\partial x_i} = 0.}$$

(5) $$\frac{\partial f(\underset{\sim}{x})}{\partial x_i} = \frac{\partial f(h(\underset{\sim}{u}), \underset{\sim}{u})}{\partial x_i} + \frac{\partial f(h(\underset{\sim}{u}), \underset{\sim}{u})}{\partial x_1}\frac{\partial h(\underset{\sim}{u})}{\partial x_i}$$

$\forall \underset{\sim}{u} \in N$

Note that $\underset{\sim}{x}_0 = (h(\underset{\sim}{u}_0), \underset{\sim}{u}_0)$ is a local extremum for $f$ subject to the constraint $g(\underset{\sim}{x}) = 0$. This implies $\underset{\sim}{u}_0$ is an unrestrained extremum for $f(h(\underset{\sim}{u}), \underset{\sim}{u})$

$$\therefore \frac{\partial f(h(\underset{\sim}{u}_0), \underset{\sim}{u}_0)}{\partial x_i} = 0$$

Hence by (5) we have.

(6)
$$\boxed{\frac{\partial f(\underset{\sim}{x_0})}{\partial x_i} + \frac{\partial f(x_0)}{\partial x_n} \cdot \frac{\partial h(\underset{\sim}{u_0})}{\partial x_n} = 0.}$$

$$\begin{pmatrix} \frac{\partial g(\underset{\sim}{x_0})}{\partial x_i} & \frac{\partial g(\underset{\sim}{x_0})}{\partial x_n} \\ \frac{\partial f(\underset{\sim}{x_0})}{\partial x_i} & \frac{\partial f(\underset{\sim}{x_0})}{\partial x_n} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\partial h(\underset{\sim}{u_0})}{\partial x_i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This implies

$$\begin{vmatrix} \dfrac{\partial g(\mathbf{x}_0)}{\partial x_i} & \dfrac{\partial g(\mathbf{x}_0)}{\partial x_1} \\[2ex] \dfrac{\partial f(\mathbf{x}_0)}{\partial x_i} & \dfrac{\partial f(\mathbf{x}_0)}{\partial x_i} \end{vmatrix} = 0.$$

$$\begin{vmatrix} \dfrac{\partial f(\mathbf{x}_0)}{\partial x_i} & \dfrac{\partial g(\mathbf{x}_0)}{\partial x_i} \\[2ex] \dfrac{\partial f(\mathbf{x}_0)}{\partial x_1} & \dfrac{\partial g(\mathbf{x}_0)}{\partial x_1} \end{vmatrix} = 0.$$

This implies the following eq$^n$ has a non-trivial sol$^n$

$$\begin{pmatrix} \frac{\partial f}{\partial x_i} & \frac{\partial g}{\partial x_i} \\ \frac{\partial f}{\partial x_1} & \frac{\partial g}{\partial x_1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since, $\frac{\partial g(\underset{\sim}{x_0})}{\partial x_1} \neq 0$, $u$ is non-zero.

In particular, taking $u=1$ we have.

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial g}{\partial x_1} \\ \frac{\partial f}{\partial x_1} & \frac{\partial g}{\partial x_1} \end{bmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial f(\underline{x}_0)}{\partial x_1} + v \frac{\partial g(\underline{x}_0)}{\partial x_1} = 0.$$

By defn of $\lambda$

$$v = -\lambda.$$

$\therefore$

$$\begin{pmatrix} \frac{\partial f}{\partial x_i} & \frac{\partial g}{\partial x_i} \\ \frac{\partial f}{\partial x_i} & \frac{\partial g}{\partial x_i} \end{pmatrix} \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$

$$\frac{\partial f(x_0)}{\partial x_i} - \lambda \frac{\partial g(x_0)}{\partial x_i} = 0.$$

Ex 1. Find a pt $(x_0, y_0)$ lying on the line $ax+by=d$ nearest to a given pt. $(x_1, y_1)$

Soln We need to minimize

$$f(x,y) = (x-x_1)^2 + (y-y_1)^2$$

subject to the constraint

$$ax+by-d=0.$$

$$L = f(x,y) - \lambda g(x,y)$$

$$= (x-x_1)^2 + (y-y_1)^2 - \lambda(ax+by-d).$$

$$\frac{\partial L}{\partial x} = 0 \quad \& \quad \frac{\partial L}{\partial y} = 0.$$

$$\left.\begin{aligned} 2(x-x_1) - a\lambda &= 0. \\ 2(y-y_1) - b\lambda &= 0 \end{aligned}\right\} \quad \begin{aligned} x_0 &= x_1 + \tfrac{1}{2}a\lambda. \\ y_0 &= y_1 + \tfrac{1}{2}b\lambda. \end{aligned}$$

$$a x_0 + b y_0 - d = 0.$$

$$a\left\{x_1 + \frac{1}{2} a \lambda\right\} + b\left\{y_1 + \frac{1}{2} b \lambda\right\} = d.$$

$$\frac{1}{2}\left(a^2 + b^2\right)\lambda = d - a x_1 - b y_1$$

$$\therefore \lambda = \frac{2\left(d - a x_1 - b y_1\right)}{a^2 + b^2}.$$

$$\left\{(x_0 - x_1)^2 + (y_0 - y_1)^2\right\}^{2}$$

$$= \left\{\frac{1}{4} a^2 \lambda^2 + \frac{1}{4} b^2 \lambda^2\right\}^{1/2}$$

$$= \left\{\frac{1}{4}\left(a^2 + b^2\right) \cdot \lambda^2\right\}^{1/2} = \frac{|d - ax_1 - by_1|}{\sqrt{a^2 + b^2}}$$

Ex2 Find the extreme value of

$f(x) = 2x+y$ subject to $x^2+y^2=4$.

Ex3 Find the pt. in the plane

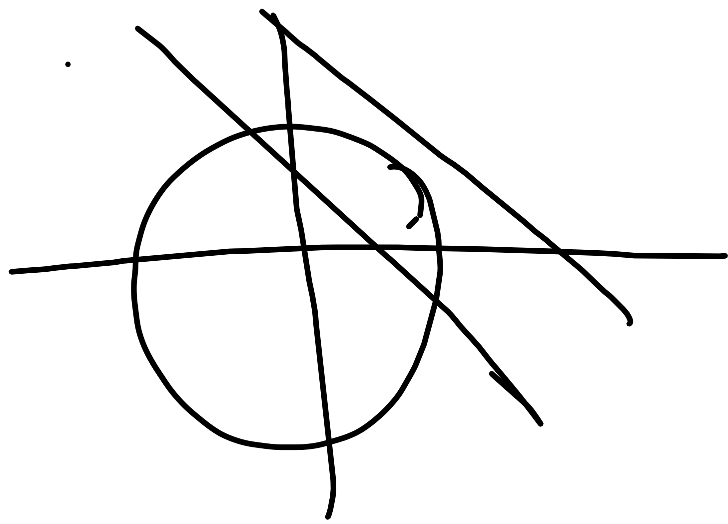
$3x+4y+z=1$.

closest to $(-1,1,1)$.

Ex 4

(a) Find the extreme values of $\sum_{i=1}^{n} x_i$ subject to $\sum_{i=1}^{n} x_i^2 = 1$

(b) Find the min value of $\sum_{i=1}^{n} x_i^2$ subject to $\sum_{i=1}^{n} x_i = 1$.



Sol$^n$ (b).

$$L = \sum x_i^2 - \lambda\left(\sum x_i - 1\right).$$

$$\frac{\partial L}{\partial x_i} = 2x_i - \lambda = 0.$$

$$x_i = \frac{1}{2}\lambda.$$

$$\frac{1}{2} n \lambda = 1. \quad \lambda = \frac{2}{n}.$$

$$\therefore \; x_i = \frac{1}{n}.$$

Ex 5 Show that

$$x^{1/p}\, y^{1/q} \le \frac{x}{p} + \frac{y}{q}, \quad x, y \ge 0.$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 0$ & $q > 0$.