Crossings in Geometric Hypergraphs

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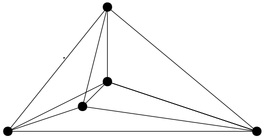
A brief history of crossing number

It started with Turan's brick factory problem which asks for the minimum number of crossings in a drawing of a complete bipartite graph $K_{n,n}$. Pál Turan asked this question during World War *II*.

In a good drawing of a graph vertices are mapped to points in general position (i.e., no three are co-linear) in $\mathbb{R}^2$ and edges are drawn as simple continuous arcs connecting vertices. A good drawing is rectilinear if edges are straight line segments.

Crossing number(Rectilinear) of a graph G, denoted by $cr(G)$ $(\overline{cr}(G))$, is minimum number of crossings among all good (Rectilinear) drawings of it.



On Structured Graphs

- A simple connected planar graph can have at most $3|V| - 6$ edges (Euler's Formula).
- In a planar drawing of any simple graph with n vertices and $e \geq 4n$ edges there are at least $\dfrac{e^3}{64n^2}$ crossing pairs of edges. (Ajtai et al., 1982)
- We know improved bounds for graphs with special structures.
  - Zarankiewicz (Zarankiewicz, 1955) conjectured that $cr(K_{m,n}) = \lfloor m/2 \rfloor \lfloor (m-1)/2 \rfloor \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$. Kleitman (Kleitman, 1970) proved Zarankiewicz's conjecture for $m \leq 6$ and an arbitrary n.
  - $0.37997 \binom{n}{4} + \Theta(n^3) \leq \overline{cr}(K_n) \leq 0.380449186 \binom{n}{4} + \Theta(n^3)$. (Ábrego et al., 2012), (Aichholzer et al., 2020)

On Structured Graphs

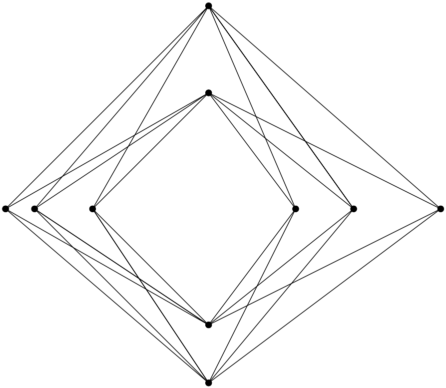


Figure: An optimal drawing of $K_{6,4}$

Generalization to Geometric Hypergraphs

A d-dimensional geometric d-hypergraph is a pair (V, E), where V (Vertex set) is a set of points in general position in $\mathbb{R}^d$ and E (Set of hyperedges) is a collection of $(d-1)$-simplices spanned by some d tuples of V. (Dey and Pach, 1998)

It is also known as d-dimensional rectilinear drawing of d-uniform hypergraph in R^d. (Anshu and Shannigrahi, 2016)

A pair of hyperedges are said to be crossing if they are vertex disjoint and contain a common point in their relative interior. (Dey and Pach, 1998)

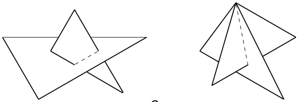


Figure: (left) crossing simplices in $\mathbb{R}^3$, (right) intersecting simplices in $\mathbb{R}^3$

Hypergraph Results

- The *d-dimensional rectilinear crossing number* of a hypergraph H, denoted by $\overline{cr}_d(H)$, is the minimum number of crossing pairs of hyperedges among all d-dimensional rectilinear drawings of H (2016, Anshu and Shannigrahi). Similarly, maximum d-dimensional rectilinear crossing number of H can be defined.
- The extremal results corresponding to planarity, were generalized. If $\overline{cr}_3(H) = 0$, then $|E| < 3n^2/2$ (Dey and Edelsbrunner, 1994). Dey and Pach proved that if $\overline{cr}_d(H) = 0$, then $|E| = O(n^{d-1})$ (1998).
- For every n and $|E| > cn^{d-1}$, $c_1 \frac{|E|^{d+1}}{n^{d(d-1)}} \leq x^d_{2,d} \leq c_2 \frac{|E|^{2+1/\lfloor d/2 \rfloor}}{n^{d/\lfloor d/2 \rfloor}}$ (Dey and Pach, 1998)
- The first lower bound $\Omega(2^d \log d/\sqrt{d})$ on $\overline{cr}_d(K^d_{2d})$ was proposed by Anshu and Shannigrahi in 2016. Currently, the best known lower bound is $\Omega(d2^d)$. This also implies $\overline{cr}_d(K^d_n) = \Omega(d2^d)\binom{n}{2d}$.

Our Results

- We improve the lower bound on $\overline{cr}_d(K_{2d}^d)$ to $\Omega\left(\frac{(4\sqrt{2}/3^{3/4})^d}{d}\right)$ which is approximately $\Omega\left(\frac{2.481^d}{d}\right)$. Note that this bound is exponentially better than previous bounds.
- The new bound implies that $\overline{cr}_d(K_n^d) = \Omega\left(\frac{2.481^d}{d}\right)\binom{n}{2d}$ since each set of $2d$ vertices creates distinct crossing pairs of hyperedges.
- The maximum number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of $K_{d\times n}^d$ is $(2^{d-1}-1)\binom{n}{2}^d$.

Our Results

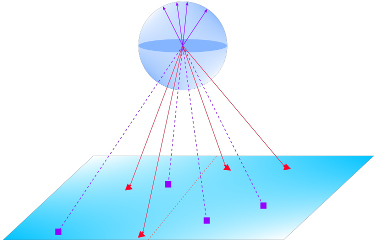
- Let c_d^m be the number of crossing pairs of hyperedges in a d-dimensional convex drawing of K_{2d}^d where all of its vertices are placed on the d-dimensional moment curve. The value of c_d^m is

$$c_d^m = \begin{cases} \binom{2d-1}{d-1} - \sum_{i=1}^{\frac{d}{2}} \binom{d}{i}\binom{d-1}{i-1} & \text{if } d \text{ is even} \\ \binom{2d-1}{d-1} - 1 - \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \binom{d-1}{i}\binom{d}{i} & \text{if } d \text{ is odd} \end{cases}$$

- The maximum number of crossing pairs of hyperedges in a 3-dimensional rectilinear drawing of K_n^3 is $3\binom{n}{6}$. The maximum number of crossing pairs of hyperedges in a 4-dimensional rectilinear drawing of K_n^4 is $13\binom{n}{8}$. These bounds are achieved when vertices are on the 3- and 4-dimensional moment curve, respectively.

Gale transform and Gale diagram

- Gale transform $G(P)$ (Gale, 1963) of a set P having $n \geq d+1$ points in R^d (whose affine hull is R^d) is a set of n vectors in R^{n-d-1}.
- It preserves the combinatorial information about the point set. Here is a Gale transform and an affine Gale diagram of 8-points in R^4.



Properties of Gale Transform

We list the properties of Gale transformation [1]

- A sequence of vectors $G = \langle g_1, g_2, \ldots, g_n \rangle$ in $\mathbb{R}^{n-d-1}$ is a Gale transform of some $P \subset \mathbb{R}^d$ if and only if G spans $\mathbb{R}^{n-d-1}$ and $\sum_{i=1}^{n} g_i = \vec{0}$.
- Points in P are in general position in $\mathbb{R}^d$ if and only if every $n-d-1$ vectors in $G(P)$ span $\mathbb{R}^{n-d-1}$.
- For $t \leq d$, consider a tuple $(i_1, i_2, \ldots, i_t)$, where $1 \leq i_1 < i_2 < \ldots < i_t \leq n$. A t-element subset $P' = \{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\} \subset P$ forms a $(t-1)$-dimensional face of $Conv(P)$ if and only if the relative interior of the convex hull of the points in $G(P) \setminus \{g_{i_1}, g_{i_2}, \ldots, g_{i_t}\}$ contains the origin.
- Consider a tuple $(i_1, i_2, \ldots, i_k)$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq m$. The convex hull of $\{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$ crosses the convex hull of $P \setminus \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$ if and only if there exists a linear separation of the vectors in $D(P)$ into $\{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$ and $D(P) \setminus \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$.

[1] J. Matoušek, Lectures in Discrete Geometry.

Other Useful Lemmas

Lemma

[2] *Let C' be a set containing $d+4$ points in general position in $\mathbb{R}^d$. There exist at least $\lfloor (d+4)/2 \rfloor$ pairs of disjoint subsets $\{C'_{i1}, C'_{i2}\}$ of C' for each i satisfying $1 \le i \le \lfloor (d+4)/2 \rfloor$ such that the following properties hold.*

1. $C'_{i1} \cup C'_{i2} = C'$ *and* $|C'_{i1}|, |C'_{i2}| \ge \lfloor (d+2)/2 \rfloor$
2. $(|C'_{i1}|-1)$*-simplex* $Conv(C'_{i1})$ *crosses the* $(|C'_{i2}|-1)$*-simplex* $Conv(C'_{i2})$ *(i.e.,* $C'_{i1} \cap C'_{i2} = \emptyset$ *and* $Conv(C'_{i1}) \cap Conv(C'_{i2}) \neq \emptyset$*).*
3. *There exist* $C''_{i1} \subseteq C'_{i1}$ *and* $C''_{i2} \subseteq C'_{i2}$ *such that* $|C''_{i1}|, |C''_{i2}| \ge \lfloor (d+2)/2 \rfloor, |C''_{i1}| + |C''_{i2}| = d+2$ *and* $(|C''_{i1}|-1)$*-simplex* $Conv(C''_{i1})$ *crosses the* $(|C''_{i2}|-1)$*-simplex* $Conv(C''_{i2})$*.*

[2] Gangopadhyay and Shannigrahi, 2020

Other Useful Lemmas

Lemma

[3] *Let a set C contain $2d$ points in general position in $\mathbb{R}^d$. Let $C' \subset C$ be a subset such that $|C'| = d + 4$. Let C'_1 and C'_2 be two disjoint subsets of C' such that $|C'_1| = c'_1, |C'_2| = c'_2, C'_1 \cup C'_2 = C'$ and $c'_1, c'_2 \geq \lfloor (d+2)/2 \rfloor$. If the $(c'_1 - 1)$-simplex formed by C'_1 crosses the $(c'_2 - 1)$-simplex formed by C'_2, then the $(d-1)$-simplex formed by some point set $B'_1 \supset C'_1$ and the $(d-1)$-simplex formed by some point set $B'_2 \supset C'_2$ satisfying $B'_1 \cap B'_2 = \emptyset$, $|B'_1|, |B'_2| = d$ and $B'_1 \cup B'_2 = C$ also form a crossing pair.*

Note: Consider a set of $d+4$ points from a set of $2d$ points. We know there exist $\Omega(d)$ crossing pairs of $(k-1)$- and $(l-1)$-simplices where $k, l \geq \lfloor (d+2)/2 \rfloor$ spanned by $d+4$ points. We can extend each of these crossing pairs of lower dimensional simplices to $\Theta(2^d/\sqrt{d})$ crossing pairs of $(d-1)$-simplices, therefore forming at least $\Omega(2^d\sqrt{d})$ crossing pairs of $(d-1)$-simplices.

[3] Gangopadhyay and Shannigrahi, 2020

Proof Outline

Lemma

Let σ be a $\lfloor d/2 \rfloor$-simplex, and τ be a $(d-1)$-simplex such that all the $d + \lfloor d/2 \rfloor + 1$ points of $Vert(\sigma) \cup Vert(\tau)$ are in general position in $\mathbb{R}^d$. At most $O((3^{3/4}/\sqrt{2})^d/\sqrt{d})$ $\lceil d/2 \rceil$-faces of τ cross σ.

- Let τ be a $(d-1)$-simplex that crosses a $\lfloor d/2 \rfloor$-simplex σ. Vertex set of τ and σ are disjoint and all vertices are in general position in R^d. We want an upper bound on the number of $\lceil d/2 \rceil$-sub simplices of τ that cross σ.
- Project to the orthogonal space of σ, call it $\sigma^\perp$. All the vertex of τ will have distinct image and σ maps to a single point O. O and the image of the vertices of τ form a totally cyclic vector configuration.
- If a $\lceil d/2 \rceil$-sub simplex of τ cross σ then the convex hull of corresponding $\lceil d/2 \rceil + 1$ vertices contains O.
- Then, apply Gale transform and Upper Bound Theorem to obtain an upper bound of $\approx \binom{3d/4}{d/2}$.

Proof Outline for Improved Lower Bound

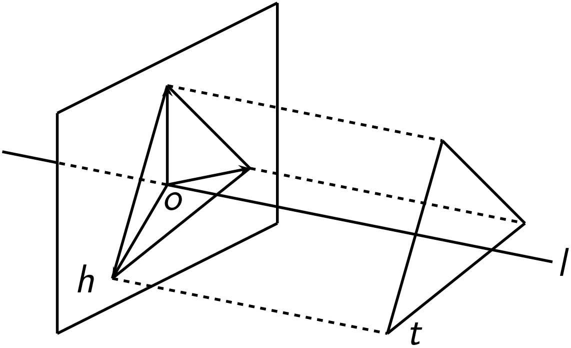


Figure: Projection argument for $d = 3$

Improvement on the Bound

- There are $\binom{2d}{d+4} = \Theta(4^d/\sqrt{d})$ ways to choose a subset of size $d+4$.
- Each of them can produce $2^d\sqrt{d}$ crossing pairs of $(d-1)$-simplices.
- Note that a crossing pair of simplices can originate from many such $(d+4)$-sized subset. We want an Upper bound on that number.
- The above mentioned lemma gives the following upper bound. $$2\binom{d}{\lceil d+2/2 \rceil} O((3^{3/4}/\sqrt{2})^d/\sqrt{d}) \times O(d^2) = O((3^{3/4}\sqrt{2})^d d)$$
- This implies that there exist at least $$\frac{\Omega\left(2^d\sqrt{d}\right)\Theta\left(4^d/\sqrt{d}\right)}{O((3^{3/4}\sqrt{2})^d d)} = \Omega\left(\frac{(4\sqrt{2}/3^{3/4})^d}{d}\right)$$ crossing pairs of hyperedges.

Important Lemmas

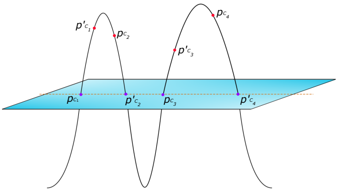


Figure: Non-crossing pair of hyperedges of $K^4_{4\times 2}$.

Lemma

(Akiyama et al., 1989) Let us consider d pairwise disjoint sets in $\mathbb{R}^d$, each consisting of two points, such that all $2d$ points are in general position. Then there exist 2 pairwise disjoint (d-1) simplices such that each simplex has one vertex from each set.

Important lemmas

Lemma

(Breen, 1973) Let $p_1 \prec p_2 \prec \ldots \prec p_{\lfloor \frac{d}{2} \rfloor + 1}$ *and* $q_1 \prec q_2 \prec \ldots \prec q_{\lceil \frac{d}{2} \rceil + 1}$ *be two distinct point sequences on the d-dimensional moment curve such that* $p_i \neq q_j$ *for any* $1 \leq i \leq \lfloor \frac{d}{2} \rfloor + 1$ *and* $1 \leq j \leq \lceil \frac{d}{2} \rceil + 1$*. The* $\lfloor \frac{d}{2} \rfloor$*-simplex and the* $\lceil \frac{d}{2} \rceil$*-simplex, formed respectively by these point sequences, cross if and only if every interval* (q_j, q_{j+1}) *contains exactly one* p_i *and every interval* (p_i, p_{i+1}) *contains exactly one* q_j*.*

Lemma

(Dey et al., 1998) Let P and Q be two vertex-disjoint $(d-1)$*-simplices such that each of the* $2d$ *vertices belonging to these simplices lies on the d-dimensional moment curve. If P and Q cross, then there exist a* $\lfloor \frac{d}{2} \rfloor$*-simplex* $U \subsetneq P$ *and another* $\lceil \frac{d}{2} \rceil$*-simplex* $V \subsetneq Q$ *such that U and V cross.*

Proof Idea

- The maximum d-dimensional rectilinear crossing number of $K^d_{d\times 2}$ is $2^{d-1}-1$.
- Let A be a set of d vertices of $K^d_{d\times 2}$ such that each vertex of A is from different parts of $K^d_{d\times 2}$. Let B be the set of rest of the vertices of $K^d_{d\times 2}$. Note that $|B| = d$ and each vertex of B is from different parts of $K^d_{d\times 2}$. The number of unordered pairs $\{A, B\}$ is $\frac{1}{2}2^d = 2^{d-1}$.
- Lemma by Akiyama et al. implies that in any d-dimensional rectilinear drawing of $K^d_{d\times 2}$, there exists a pair of disjoint simplices such that each simplex has one vertex from each part of $K^d_{d\times 2}$. This implies the maximum number of unordered pairs $\{A, B\}$ such that $(d-1)$-simplex formed by the vertices of A forms a crossing with the $(d-1)$-simplex formed by the vertices of B is $2^{d-1}-1$.
- For each i satisfying $1 \leq i \leq d$, let us denote the i^{th} part of the vertex set of $K^d_{d\times 2}$ by C_i. Let $\{p_{c_i}, p'_{c_i}\}$ denote the set of 2 vertices in C_i. In this particular drawing, the vertices of $K^d_{d\times 2}$ are placed on the d-dimensional moment curve such that they satisfy the following ordering on the d-dimensional moment curve. $p_{c_1} \prec p'_{c_1} \prec p_{c_2} \prec p'_{c_2} \ldots \prec p_{c_{d-1}} \prec p'_{c_{d-1}} \prec p_{c_d} \prec p'_{c_d}$.

Proof Idea

- Lemmas by Breen and Dey et al. along with pigeonhole principle imply that there exists only one pair of hyperedges $\{A, B\}$ that does not form a crossing where $\{A, B\}$ are

$$A = \{p_{c_1}, p'_{c_2}, p_{c_3}, p'_{c_4}, \ldots, p_{c_{d-1}}, p'_{c_d}\}$$

$$B = \{p'_{c_1}, p_{c_2}, p'_{c_3}, p_{c_4}, \ldots, p'_{c_{d-1}}, p_{c_d}\}$$

.

- For each i satisfying $1 \leq i \leq d$, let C_i be the i^{th} color class of the vertex set of $K^d_{d\times n}$. Let $\{p^i_1, p^i_2, \ldots, p^i_n\}$ be the set of n vertices in C_i. Consider the following arrangement of the vertices of $K^d_{d\times n}$ on the d-dimensional moment curve.
  - Any vertex of C_i precedes any vertex of C_j if $i < j$.
  - For each i satisfying $1 \leq i \leq d$, $p^i_l \prec p^i_m$ if $l < m$.
- In this arrangement, any $K^d_{d\times 2}$ contains $2^{d-1} - 1$ crossing pairs of hyperedges. Thus, the maximum number of crossing pairs of hyperedges in a d-dimensional rectilinear drawing of $K^d_{d\times n}$ is $(2^{d-1} - 1)\binom{n}{2}^d$.

Open Questions

- Is the convex hull of the crossing optimal drawing of K_n^d is a d-simplex?
- Does all neighborly polytope (with $2d$ vertices in general position) produces the same number of crossing pairs of hyperedges as the cyclic d-polytopes? The exact number of crossing pairs of hyperedges are known if the d-dimensional rectilinear drawing of K_{2d}^d (And therefore K_n^d) is a cyclic d-polytope.
- Is this number, i.e., $c_d^m\binom{n}{2d}$ is the maximum for $d > 4$?
- There is a significant gap between the lower bound and the upper bound on the d-dimensional rectilinear crossing number of K_{2d}^d. Can we reduce this (at least for smaller values of d)?

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