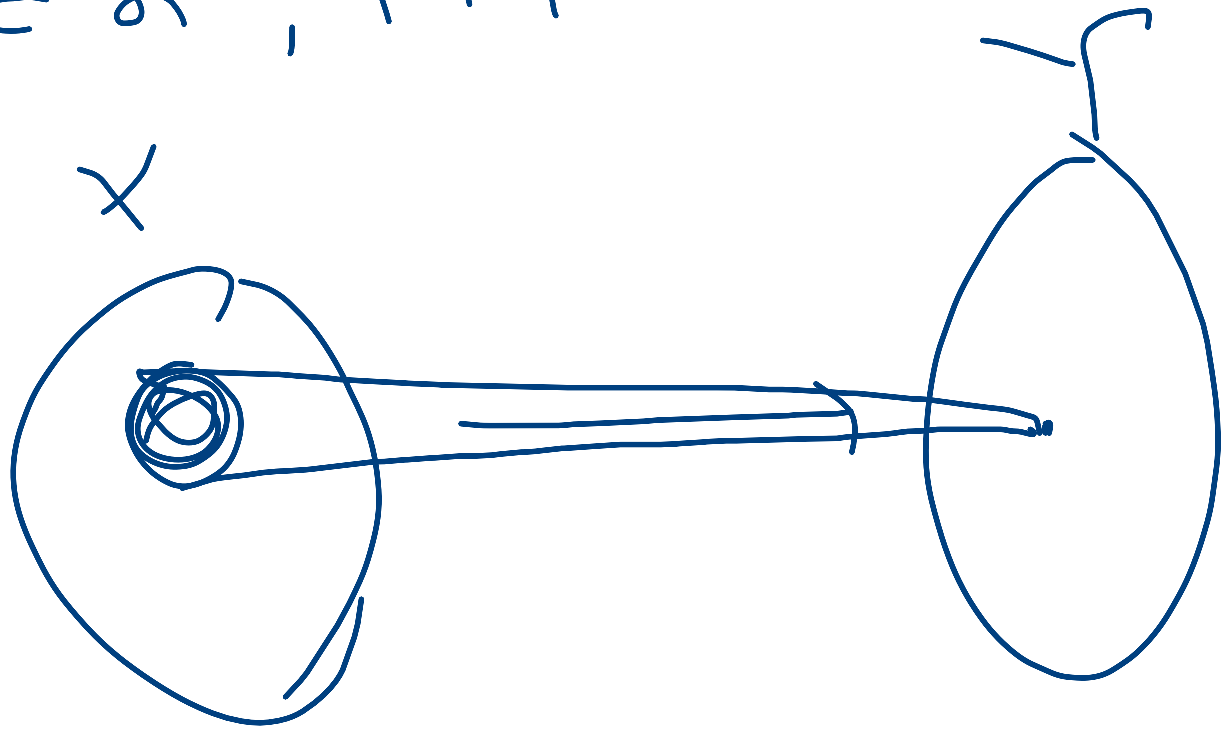
Partition Numbers

Def^n Let $p(n,k)$ denote the no. of ways of writing n as sum of k +ve integers

$p(4,4)=1$; $1+1+1+1$; $p(4,2)=2$; $1+3$; $2+2$

$p(4,3)=1$; $1+1+2$ $p(4,1)=1$

Note that $p(n,k)$ also denote the no. of onto $\underline{fns}$: $f: X \longrightarrow Y$, where $|X| = n$, $|Y| = k$ $\underline{\text{and}}$ X and Y are unlabelled



$$p(n) = \sum_{k=1}^{n} p(n,k), \quad p(0) = 1.$$

$$p(4) = 1+1+2+1 = 5$$

Defn Let $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ be a partition of n into k parts.

Assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$.

With such a partition we associate its Ferrars diagram

Which is an array of k rows of dots with the i^{th} row consisting of λ_i dots.

$$6+2+2+1+1=12$$

By interchanging the rows and the col^{ns} of a Ferrers diagram, we obtain its Transpose which is a Ferrers diagram in which the corr^{n} partition of n has k as its largest summand

Hence the no. of partitions of n into k parts equals the no. of partition of n in which the largest summand is k.

<u>Thm</u>

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Pf RHS $= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$

$= (1+x+x^2+\cdots)(1+x^2+x^{2\cdot 2}+x^{3\cdot 2}+\cdots)\cdots$

In order to find the coeff^n of x^n on the RHS assume that we pick $x^{m(k)\cdot k}$ from the kth GP series

$$\prod_k x^{m(k)k} = x^n$$

Thus we obtain the following

$$m(1).1 + m(2).2 + m(3).3 + \cdots = n \quad (1)$$

Let t be the largest integer s.t. $m(t) \neq 0$.

Then (1) reduces to

$$m(1).1 + m(2).2 + \cdots + m(t).t = n \quad (2)$$

The no. of solns to (2) equals the no. of partitions of n.

Hence the coeff$^{\underline{n}}$ of x^n on the RHS is $p(n)$

Thm

$$\sum_{n=k}^{\infty} p(n,k)\, x^n = x^k \prod_{j=1}^{k} \frac{1}{1-x^j}$$

Pf. Let $p(n, \leq k)$ denote the no. of partitions of n into <u>at most</u> k parts.

Then clearly, $p(n,k) = p(n, \leq k) - p(n, \leq k-1)$

Claim $\sum_{n=k}^{\infty} p(n, \le k)x^n = \prod_{j=1}^{k} \frac{1}{1-x^j}$

$$RHS = (1+x+x^2+\dots)(1+x^2+(x^2)^2+\dots)(1+x^3+(x^3)^2+\dots)$$
$$\dots (1+x^k+x^{2k}+\dots)$$

The coeff$\underline{\text{n}}$ of x^n is the no. of sol$\underline{\text{ns}}$ of $m(1)1+m(2).2+\dots+m(k)k = n$, where $m(1)+m(2)+\dots+m(k) \le k$.

Hence:

$$\sum_{n=0}^{\infty} p(n) x^n = \sum_{n=k}^{\infty} p(n, \leq k) x^n - \sum_{n=k-1}^{\infty} p(n, \leq k-1) x^n$$

$$= \prod_{j=1}^{k} \frac{1}{1-x^j} - \prod_{j=1}^{k-1} \frac{1}{1-x^j}$$

$$= \prod_{j=1}^{k-1} \left\{ \frac{1}{1-x^k} - 1 \right\}$$

$$= \frac{x^k}{1-x^k} \prod_{j=1}^{k-1} = x^k \prod_{j=1}^{k} \frac{1}{1-x^j}$$

Def^n Let $p(n, O)$ denote the no. of partitions of n in which each summand is odd.

$p(6, O) = 4$: $1+1+1+1+1+1$; $1+1+1+3$; $1+5$; $3+3$

$p(n, D)$ be the no. of partitions of n into distinct parts.

Thm (Euler) $p(n, O) = p(n, D)$.

Pf The g.f. of $p(n, O)$ is

$$\sum_{n=0}^{\infty} p(n, O) x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$$

$$= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^6} \cdots$$

$$= (1+x)(1+x^2)(1+x^3)(1+x^4) \cdots$$

$$= \sum_{n=0}^{\infty} p(n, D) x^n$$

Ex Show that the no. of partitions of n into summands none of which occurs exactly once is the same as the no. of partitions of n into summands none of which is congruent to 1 or 5 modulo 6.