PIE

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \le k_1 < \cdots < k_j \le n} \left|A_{k_1} \cap \cdots \cap A_{k_j}\right|,$$

where the summation is taken over all seq$^{\underline{ns}}$ of length j in $\{1, \ldots, n\}$.

Ex 1 The number of onto f^{ns} from $X \xrightarrow{\text{onto}} Y$, where $|X| = x$ & $|Y| = y$. $= T(x, y)$.

pf. W.l.s. let $X = \{1, 2, \ldots, x\}$

$Y = \{1, 2, \ldots, y\}$.

Let $A_i = \{ f : f$ is a f^n from $X \to Y$ s.t. $f(x) \neq i \;\; \forall x \}$, $1 \leq i \leq y$.

Clearly, $|A_i| = (y-1)^x$.

In general.

$$|A_{k_1} \cap \dots \cap A_{k_j}| = (y-j)^x.$$

By PIE,

$$\left|\bigcup_{i=1}^{y} A_i\right| = \sum_{j=1}^{y} (-1)^{j+1} \sum_{1 \le k_1 < \dots < k_j \le n} |A_{k_1} \cap \dots \cap A_{k_j}|$$

$$= \sum_{j=1}^{y} (-1)^{j+1} \binom{y}{j} (y-j)^x$$

$$T(x,y) = y^x - \{ \quad \}$$

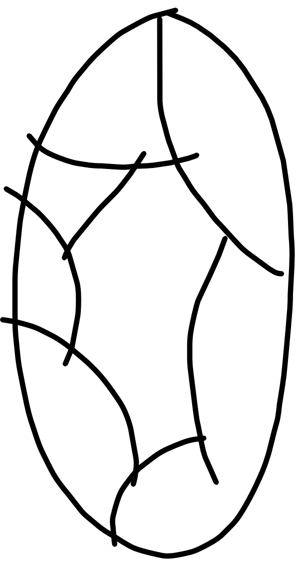
$$= \sum_{j=0}^{y} (-1)^j \binom{y}{j} (y-j)^x$$

$$= \sum (-1)^{y-j} \binom{y}{j} j^x$$

$$T(x,y) = \sum_{j=0}^{y} (-1)^{y-j} \binom{y}{j} j^x$$

Def^n Stirling's number of the 2^{nd} kind

$$S(x,y) = \frac{1}{y!} T(x,y)$$



= # of onto f^n from a set of x labeled elements onto a set of y unlabelled elements.

= # of partitions of a set of x labeled elements into y subsets.

Fix an element $y^* \in Y$. X

Clearly $S(x,y) =$ # of partitions of X into y subsets in which $\{y^*\}$ is a member of the partition

+ # of partitions in which $\{y^*\}$ is not a member of the partition.

$$\boxed{S(x,y) = S(x-1,y-1) + y\,S(x-1,y)}$$

Ex Euler's φ-fn.

$$\varphi(n) = \begin{cases} 1 & \text{if } n = 1 \\ \text{\# of +ve integers} < n \text{ that are co-prime to } n, & \text{if } n > 1 \end{cases}$$

$$\varphi(6) = 2$$

$1 \quad \cancel{2} \quad \cancel{3} \quad \cancel{4} \quad 5 \quad 6$

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be the factorization of n.

Then $\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right)$.

Pf. Let $A_i = \{ m \le n : p_i \mid m \}$.

$|A_i| = \frac{n}{p_i}$

In general.

$$\left|A_{k_1}\cap \cdots \cap A_{k_j}\right| = \frac{n}{p_{k_1}p_{k_2}p \cdots p_{k_j}}$$

By PIE,

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{j=1}^{k} (-1)^{j+1} \sum_{1\le k_1 < \cdots < k_j \le k} \left|A_{k_1}\cap \cdots \cap A_{k_j}\right|$$

$$= \sum_{j=1}^{k} (-1)^{j+1} \sum_{1\le k_1 < \cdots < k_j \le k} \frac{n}{p_{k_1}\cdots p_{k_j}}$$

$$\varphi(n) = n - \sum \ldots$$

$$= \sum_{j=0}^{r} (-1)^j \sum_{1 \le k_1 < \cdots < k_j \le r} \frac{n}{p_{k_1} \cdots p_{k_j}}$$

$$= n \sum_{j=0}^{r} (-1)^j \sum_{1 \le k_1 < \cdots < k_j \le r} \frac{1}{p_{k_1} \cdots p_{k_j}}$$

claim

$$= n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right) \quad (Ex)$$

Ex # of derangements / Hatcheck problem.

$$S_n = \{ \pi : \{1, \dots, n\} \to \{1, \dots, n\}, \; \pi \text{ bijection} \}$$

$$D_n = \{ \pi \in S_n : \pi(i) \neq i, \forall i \}.$$

Let $A_i = \{ \pi \in S_n : \pi(i) = i \}.$

$$|A_i| = (n-1)!$$

By PIE,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{j=1}^{n} (-1)^{j+1} \sum_{1\le k_1 < \cdots < k_j \le n} \frac{|A_{k_1} \cap \cdots \cap A_{k_j}|}{}$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} (n-j)!$$

$$\therefore |D_n| = D_n = n! + \sum_{j=1}^{n} (-1)^{j} \binom{n}{j} (n-j)!$$

$$= \sum_{j=1}^{n} (-1)^{j} \binom{n}{j} (n-j)! = \sum_{j=0}^{n} (-1)^{j} \frac{n!}{j!}$$

$$= n! \sum_{j=0}^{n} (-1)^j \frac{1}{j!}$$

$$= n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + - \cdots + (-1)^n \frac{1}{n!} \right\}.$$

Ex. Show that the expected no. of fixed pts is $\underline{1}$.

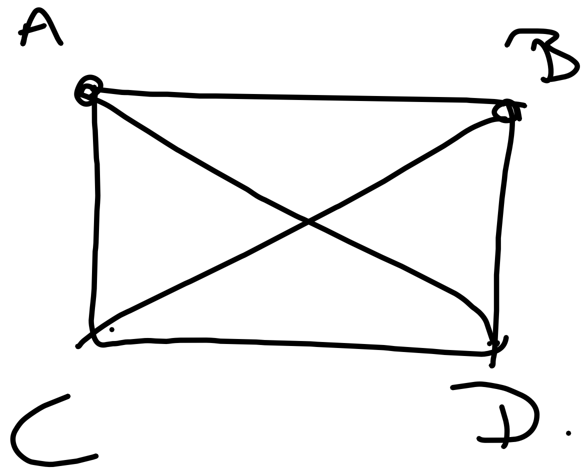
Chromatic Polyⁿ.

Given a graph $G = (V, E)$ of n vertices, and a +ve integer x, let

$$\chi(G, x) = \#\text{ of proper colourings of the vertices in } V \text{ using at most } x \text{ colors.}$$

χ is called the ~~chro~~ chromatic polyⁿ of G.

Ex 1 G is K_4.



$$\chi(K_4, x) = x(x-1)(x-2)(x-3)$$

A can be coloured in x ways.
then B can be coloured in $(x-1)$
C - - - - . $(x-2)$
D - - - $(x-3)$

$G = (V, E)$, $|V| = n$
$|E| = r$.

Let $E = \{e_1, \ldots, e_r\}$.

$A_i = \{c : c$ is a colouring of G s.t. the vertices of e_i receive same colour$\}$.

$\bigcup(A_i)$

By PIE.

$$\left| \bigcup_{i=1}^{r} A_i \right| = \sum_{j=1}^{r} (-1)^{j+1} \sum_{1 \le k_1 < \cdots < k_j \le r} \boxed{\left| A_{k_1} \cap \cdots \cap A_{k_j} \right|} \quad (1)$$

$x^{c(A)}$

Hence the no. of proper colouring is

$$= x^n + \sum_{j=1}^{r} (-1)^j \sum_{1 \le k_1 < \cdots < k_j \le r} \left| A_{k_1} \cap \cdots \cap A_{k_j} \right|. \quad (2)$$

Note that $A_{k_1} \cap \cdots \cap A_{k_j}$ is the set of all colourings of G in which each edge e_{k_i} receive same colours.

Let H be a spanning subgraph of G consisting of all vertices of G with edges $e_{k_1}, e_{k_2}, \ldots, e_{k_j}$.

Note colouring G s.t. each edge e_{x_i} receive same colours is the same as colouring H s.t. each edge of e_{x_i} receive same colour. Let $c(H)$ be the no. of components of H. Then the no. of ways of colouring H so that each edge e_{x_i} receive same colour

$$= x^{c(H)}$$

Hence (2) can be written as.

$$x^n + \sum_{j=1}^{b} (-1)^j x^{c(H)}, \text{ where}$$

$c(H)$ is the no. of connected components of a spanning subgraph of G consisting of edges $e_{k_1}, e_{k_2} \ldots e_{k_j}$

$$\sum_{j,c} (-1)^j x^{h(j,c)}$$