Two corrections

Ex If a graph G has $\boxed{2n}$ vertices & n^2+1 edges, then it has a $\triangle$

G

H : spanning subgraph of G with j edges & c components.

# of ways of coloring H is x^c

Now $h(j, c) =$ # of spanning subgraphs of G with c components.

$$\chi(G, x) = \sum_{j,c} (-1)^j \, h(j, c)\, x^c.$$

PIE:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{j=0}^{n} (-1)^{j+1} \sum_{1\le k_1<\dots<k_j\le n} \left|A_{k_1}\cap\dots\cap A_{k_j}\right|$$

Ex Let (Ω, P) be a prob. space, where Ω is finite.
Let $E_1, E_2, \dots, E_n$ be events on Ω.

Then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=0}^{n} (-1)^{j+1} \sum_{1\le k_1<\dots<k_j\le n} P\left(A_{k_1}\cap\dots\cap A_{k_j}\right)$$

Def^n Given a seq$^{\underline{n}}$ $\{a_n : n \geq 0\}$

we define the Ordinary generating f^n by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

The exponential generating $f^{\underline{n}}$ of the seq$^{\underline{n}}$ $\{a_n\}$ is given by

$$\boxed{g(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}}$$

Def^n A seq^n $\{a_n\}$ is said to satisfy a linear homogeneous recurrence relation with const. coeff^{ts} if

$$(1) \quad \boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_p a_{n-p}}, \text{ for all } n \geq p.$$

Ex. $F_n = F_{n-1} + F_{n-2}, \; n \geq 2$

Defn The characteristic eqn associated with the recurrence relation (1) is

$$\boxed{x^p - c_1 x^{p-1} - c_2 x^{p-2} \cdots - c_p = 0.} \quad (2)$$

$$\boxed{x^2 - x - 1 = 0}$$

Thm 1. Let the seq^n $\{a_n\}$ satisfy the recurrence relation ① & suppose $\alpha_1, \alpha_2, \ldots, \alpha_p$ are distinct roots of the characteristic eq^n ②. Then the general sol^n of ① is given by

$$\boxed{a_n = \lambda_1 \alpha_1^n + \cdots + \lambda_p \alpha_p^n}, \text{ for}$$

some constants $\lambda_1, \ldots, \lambda_p$.

Pf Let $f(x)$ be the o.g.f. of $\{a_n\}$.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{p-1} a_n x^n + \sum_{n=p}^{\infty} a_n x^n$$

$$= h(x) + \sum_{n=p}^{\infty} \left(\sum_{i=1}^{p} c_i a_{n-i} \right) x^n$$

$$= h(x) + \sum_{n=p}^{\infty} \sum_{i=1}^{p} c_i x^i a_{n-i} x^{n-i}$$

$$= h(x) + \sum_{i=1}^{p} c_i x^i \left(\sum_{n=p}^{\infty} a_{n-i} x^{n-i} \right)$$

$$= h(x) + \sum_{i=1}^{p} c_i x^i \left(\sum_{n=p-i}^{\infty} a_n x^n \right)$$

$$f(x) = h(x) + \sum_{i=1}^{p} c_i x^i \left(f(x) - a_0 - a_1 x - \dots - a_{p-i-1} x^{p-i-1} \right)$$

$$f(x)\left\{ 1 - c_1 x - c_2 x^2 \dots - c_p x^p \right\} = h(x) - \underbrace{\sum_{i=1}^{p} c_i \left(a_0 x^i + a_1 x^{i+1} \dots + a_{p-i-1} x^{p-1} \right)}_{\text{poly}^n \text{ of deg } p-1}$$

$$\boxed{f(x)\left\{1 - c_1 x - c_2 x^2 - \cdots - c_p x^p\right\} = g(x)}$$

Claim $1 - c_1 x - c_2 x^2 - c_p x^p = (1-\alpha_1 x)\cdots(1-\alpha_p x)$,

where $\alpha_1, \ldots, \alpha_p$ are the distinct roots of (2)

$$\left(1 - c_1 x - c_2 x^2 - \cdots - c_p x^p\right) = y^{-p}\left(y^p - c_1 y^{p-1} \cdots - c_p\right), \text{ where } y = \frac{1}{x}$$

$$= y^{-p}(y-\alpha_1)(y-\alpha_2)\cdots(y-\alpha_p)$$

$$= (1-\alpha_1 x)\cdots(1-\alpha_p x)$$

$$f(x) = \frac{g(x)}{(1-\alpha_1 x)\cdots(1-\alpha_p x)}$$

$$= \frac{\lambda_1}{1-\alpha_1 x} + \frac{\lambda_2}{1-\alpha_2 x} + \cdots + \frac{\lambda_p}{1-\alpha_p x}, \quad \text{where } A_i \text{ are consts.}$$

$$= \lambda_1\left(\sum_{i=0}^{\infty}(\alpha_1 x)^i\right) + \cdots + \lambda_p \sum_{i=0}^{\infty}(\alpha_p x)^i$$

n.Th term $\left(\lambda_1 \alpha_1^n + \lambda_2 \alpha_2^n + \cdots + \lambda_p \alpha_p^n\right) x^n$

Equating coeff^{s} of x^n we have

$$a_n = \lambda_1 \alpha_1^n + \cdots + \lambda_p \alpha_p^n$$

Thm 2 Suppose the characteristic eqn (2) has roots α_1 with multiplicity μ_1

α_2 - - - - μ_2

$\vdots$

α_k . - . - - . μ_k.

Then the basic sol^{ns} are

α_1^n, $n\alpha_1^{n-1}$, $n^2\alpha_1^{n-2}$ . . . , $n^{\mu_1-1}\alpha_1^{n-\mu_1+1}$

α_2^n, $n\alpha_2^{n-1}$ -

Ex $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.

$F_0 = F_1 = 1$.

The characteristic eqn is

$$x^2 - x - 1 = 0.$$

The roots are $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$

Hence the general soln is

$$F_n = \lambda_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \lambda_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$\lambda_1 + \lambda_2 = 1$

$\lambda_1\left(\frac{1+\sqrt{5}}{2}\right) + \lambda_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$\lambda_1\left\{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right\} = \frac{1-\sqrt{5}}{2} - 1$

$-\sqrt{5}\,\lambda_1 = -\frac{\sqrt{5}+1}{2}$

$\lambda_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}$

$\lambda_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$

Ex 2 A codeword is a string over the alphabet $\{0,1,2,3\}$

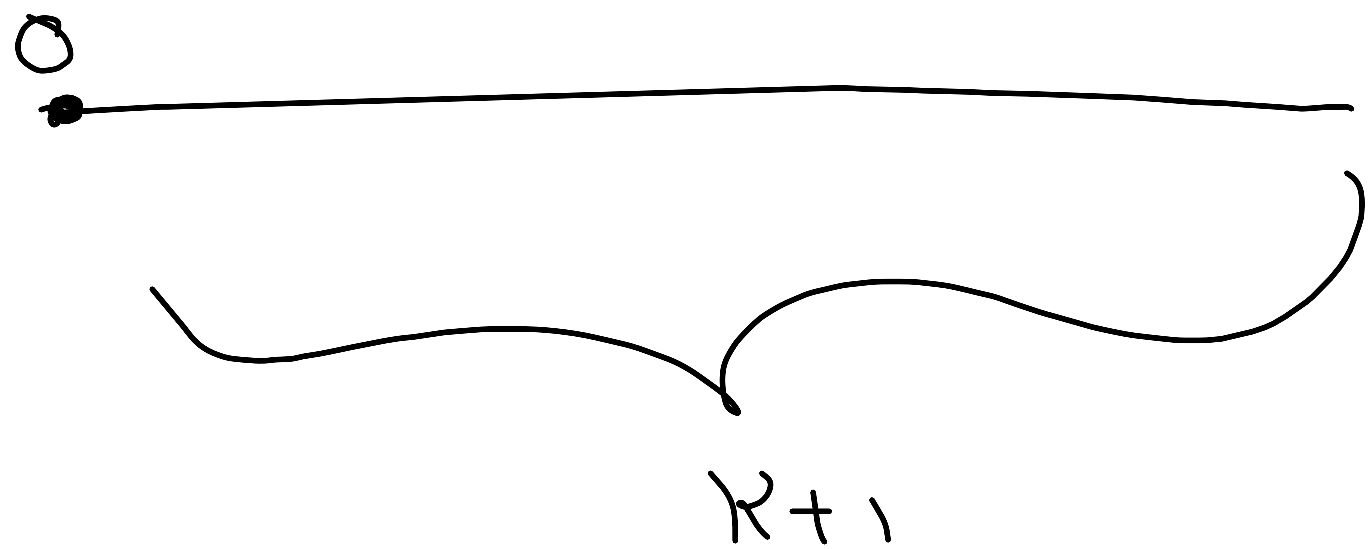
A codeword is said to be legitimate if it contains an even nos. of 0's.

Find the no. of legitimate codewords of length k. Denote it by a_k.

Take $a_0 = 1$

$$a_{k+1} = (4^k - a_k) + 3a_k = 4^k + 2a_k.$$

0 occupying 1st position



$$a_{k+1} = 4^k + 2a_k \quad ; \quad k \geq 0$$

$$f(x) = \frac{1-3x}{(1-2x)(1-4x)} = \frac{A_1}{1-2x} + \frac{A_2}{1-4x}$$

$$\sum_{k=0}^{\infty} a_{k+1} x^{k+1} = \sum_{k=0}^{\infty} 4^k x^{k+1} + 2\sum_{k=0}^{\infty} a_k x^{k+1}$$

$$f(x) - a_0 = \frac{x}{1-4x} + 2x \cdot f(x)$$

$$f(x)\left[1-2x\right] = \frac{x}{1-4x} + 1 = \frac{1-3x}{1-4x}.$$

$$\frac{A_1}{1-2x} + \frac{A_2}{1-4x} = \frac{1-3x}{(1-2x)(1-4x)}$$

$$= \quad 1$$

$$= \frac{(1-2x) - x}{(1-2x)(1-4x)}$$

$$= \frac{1}{1-4x} + \frac{-x}{(1-2x)(1-4x)}$$

$$= \frac{1}{1-4x} + \frac{(1-4x)+1}{4(1-2x)(1-4x)}$$

$$= \frac{1}{1-4x} + \frac{1}{4(1-2x)} + \frac{1}{4(1-2x)(1-4x)}$$

$$f(x) = \frac{1}{2}\left\{\frac{1}{1-2x} + \frac{1}{1-4x}\right\}.$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{\infty}(2x)^n. + \sum_{n=0}^{\infty}(4x)^n.\right\}.$$

$$a_k = \frac{1}{2}\left[2^k + 4^k\right]. \quad k \geq 0.$$

$$D_{n+1} = n\left(D_n + D_{n-1}\right).$$