Derrangements. D_n

Claim $D_{n+1} = n\left(D_n + D_{n-1}\right)$; $n > 2$.

For $n = 1$.

$$D_2 = \underset{\substack{\shortparallel \\ 0}}{D_1} + D_0$$

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We define ~~to~~ $D_0 = 1$.

<u>Claim</u> $D_{n+1} = n(D_n + D_{n-1}), \ \forall n \geq 2$.

<u>Case 1</u>.

$\underline{1}$ $\quad$ $K^{\underline{th}}$ position $\quad$ $-$ $n+1$

K $\quad$ 1

Count in this case is D_{n-1} for each choice of K. Hence Total count is $n D_{n-1}$

Case 2: k is occupying the first position but 1 does not occupy the k^{th} position

$$\begin{array}{cccccccccc} 1 & & & & & k^{th}\text{ position} & & & & \\ k & \bigcirc & - & - & - & 1 & - & \cdot & \cdot & - \end{array}$$

The count is D_n since we have a derangement of n elements

Total count is nD_n.

Hence ${}^{*}D_{n+1} = n(D_n + D_{n-1})$, $n \geq 2$.

Ex From the above, show that

$$D_{n+1} = (n+1)D_n + (-1)^{n+1}, \quad n \geq 0.$$

Let $d(x)$ be the exp^n g.f. for $\{D_n\}$

$$d(x) = \sum_{n=0}^{\infty} \frac{D_n}{n!} x^n.$$

<u>Claim</u> $(1-x)\,d'(x) = x\,d(x)$.

$$d'(x) = \sum_{n=1}^{\infty} \frac{D_n}{(n-1)!} x^{n-1}.$$

$$\therefore (1-x)\,d'(x) = \sum_{n=0}^{\infty} \frac{D_{n+1}}{n!} x^n - x \sum_{n=1}^{\infty} \frac{D_n}{(n-1)!} x^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{D_{n+1}}{n!} x^n - \sum_{n=1}^{\infty} \frac{n D_n}{n!} x^n.$$

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$$= \sum_{n=0}^{\infty} \frac{D_{n+1} - n D_n}{n!} x^n.$$

$$= \sum_{n=1}^{\infty} \frac{n D_{n-1}}{n!} x^n$$

$$= \sum_{n=1}^{\infty} \frac{D_{n-1}}{(n-1)!} x^{n-1} = d(x).$$

Hence we have.

$$(1-x)\,d'(x) = x\,d(x)$$

$$\frac{d'(x)}{d(x)} = \frac{x}{1-x} = \frac{1}{1-x} - 1.$$

$$\log d(x) = -\log(1-x) - x + \text{const}.$$

$$\log d(x)(1-x) = -x + C.$$

$$e^{-x}\cdot \underset{1}{\overset{\|}{\text{const}}} = d(x)(1-x).$$

since $D_0 = 1$

$$d(x) = e^{-x} \cdot \frac{1}{1-x}.$$

$$= \left\{ \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n!} \right\} \left(1 + x + x^2 + \cdots \right).$$

$$= \left\{ 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} \qquad \right\} \left\{ 1 + x + x^2 + \cdots \right\}$$

$$\frac{D_n}{n!} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \qquad \therefore D_n = n! \left\{ 1 - \frac{1}{1!} + \cdots + \frac{(-1)^n}{n!} \right\}$$

Def^n Given two seq^{ns} $\{a_n\}$ and $\{b_n\}$, the convolution of $\{a_n\}, \{b_n\}$ denoted by $\{a_n\} * \{b_n\}$ is the seq^n $\{c_n\}$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

Lemma Let $a(x)$, $b(x)$ be the o.s.fs of $\{a_n\}$ and $\{b_n\}$ & let $c(x)$ be the o.s.f. of $\{c_n\}$

Then $c(x) = a(x) \cdot b(x)$.

No. of ways of forming the product $a_1 \cdots a_n$, where a_i are real

$$(a_1 \cdot (a_2 (a_3 \cdot a_4))).$$

$$((a_1 \cdot a_2) \cdot (a_3 \cdot a_4))$$

$$(((a_1 \cdot a_2) \cdot a_3) \cdot a_4).$$

No. of ways of forming the product $a_1 \cdot \cdots \cdot a_n$, where a_i's are real.

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$(((a_1 \cdot a_2) \cdot a_3) \cdot a_4)$.

Let S_n denote the # of ways of obtaining the product $a_1 a_2 \dots a_n$.

$S_0 = ?$, $S_1 = 1$.

Suppose the last multiplication is at position k, $1 \le k \le n-1$

$$(a_1 \dots a_k)(a_{k+1} \dots a_n) \quad S_k \cdot S_{n-k}.$$

$$S_n = \sum_{k=1}^{n-1} S_k S_{n-k}. \quad , n > 1.$$

$$\boxed{\begin{array}{l} S_n' = S_n. \\ S_1' = 0 \end{array}}$$

Define $S_0 = 0$ & $S_1 = 1$

$$S_n = \sum_{k=0}^{n} S_k S_{n-k}.$$

Let $T_n = \{S_n\} * \{S_n\}$.

Let $f(x)$ be the o.s.f for $\{T_n\}$.

$S_n = T_n$ for $n > 1$

$$S_n' = T_n \quad \forall n.$$

$$\sum_{n=0}^{\infty} S_n' x^n = \sum_{0}^{\infty} T_n x^n.$$

$$f(x) - x = f(x) \cdot f(x)$$

$$[f(x)]^2 - f(x) + x = 0$$

$$\therefore f(x) = \frac{1 \pm \sqrt{1-4x}}{2}.$$

$\therefore \quad f(x) = \frac{1}{2}\left\{1 \pm (1-4x)^{1/2}\right\}.$

n^{th} coeff^n of $(1-4x)^{1/2}$

$$= \binom{\frac{1}{2}}{n}(-4)^n$$

$\therefore \quad f(x) = \frac{1}{2}\left\{ 1 \pm (1-4x)^{1/2} \right\}.$

n^{th} coeff^n of $\underline{(1-4x)^{1/2}}$

$$= \binom{\frac{1}{2}}{n} (-4)^n$$

$$= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!}(-4)^n$$

$$= (-1)^{2n-1}\,2^n \cdot \frac{(1\cdot 3 \cdots (2n-3))\cdot(n-1)!}{n!} \times \frac{}{(n-1)!}$$

$$= (-1)^{2n-1}\,2\,\frac{1\cdot 2\cdot 3\cdot 4 \cdots (2n-3)(2n-2)}{(n-1)!\;n!}$$

$$= 2(-1)^{2n-1}\binom{2n-2}{n-1}\frac{(n-1)!}{n!} = -\frac{2}{n}\binom{2n-2}{n-1}$$

$$\frac{1-(1-4x)^{1/2}}{2}.$$

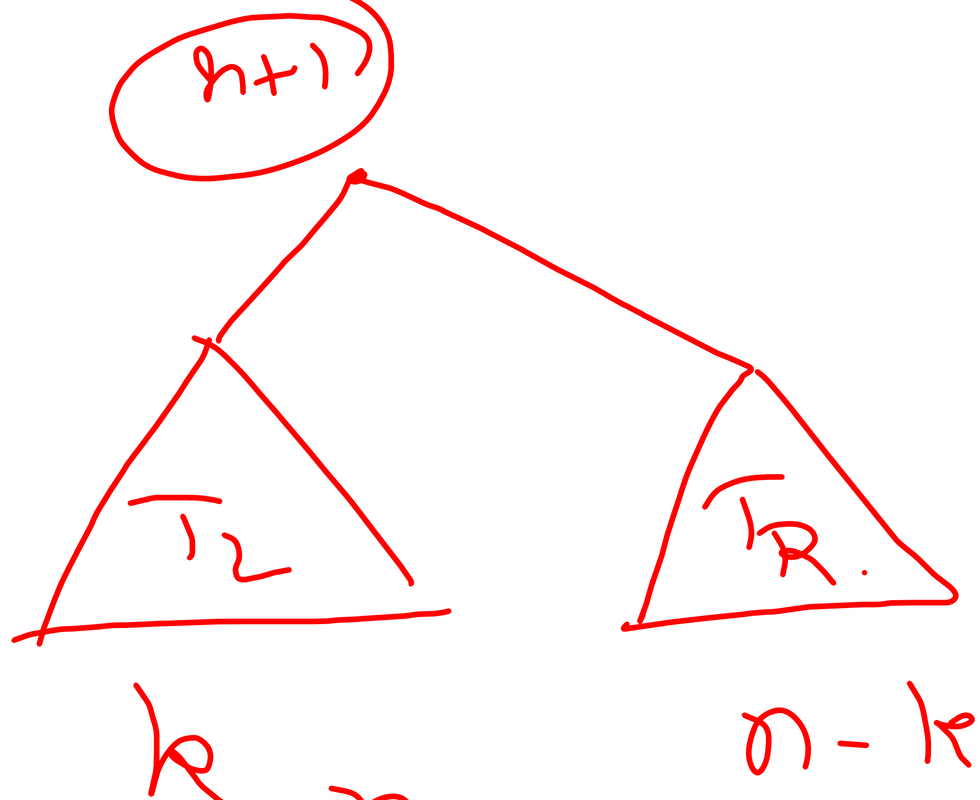
n^{th} coeff^n $= \frac{1}{n} \cdot \binom{2n-2}{n-1}$

$n+1^{st}$ coeff^n $= \frac{1}{n+1}\binom{2n}{n}$

$\downarrow$

n^{th} Catalan no.

To count the no. of simple Ordered Rooted tree.



$0 \leq k \leq n$.

$$a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k}, \quad a_0 = 1$$