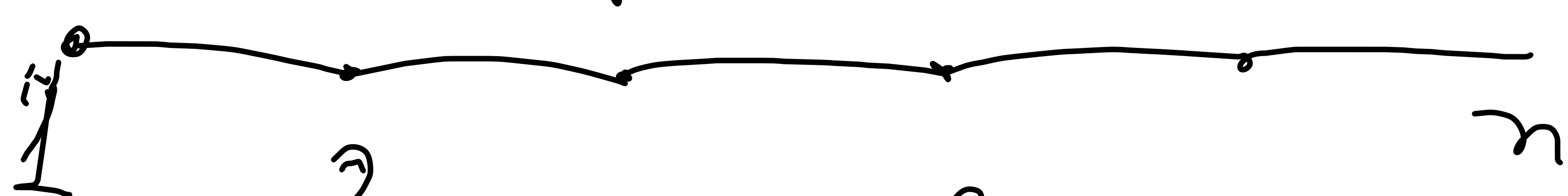
Graph automorphism.

① Set of automorphisms form a group.

$$\mathrm{Aut}(P_n) = \{ i, \sigma \}$$

Eg. P_n



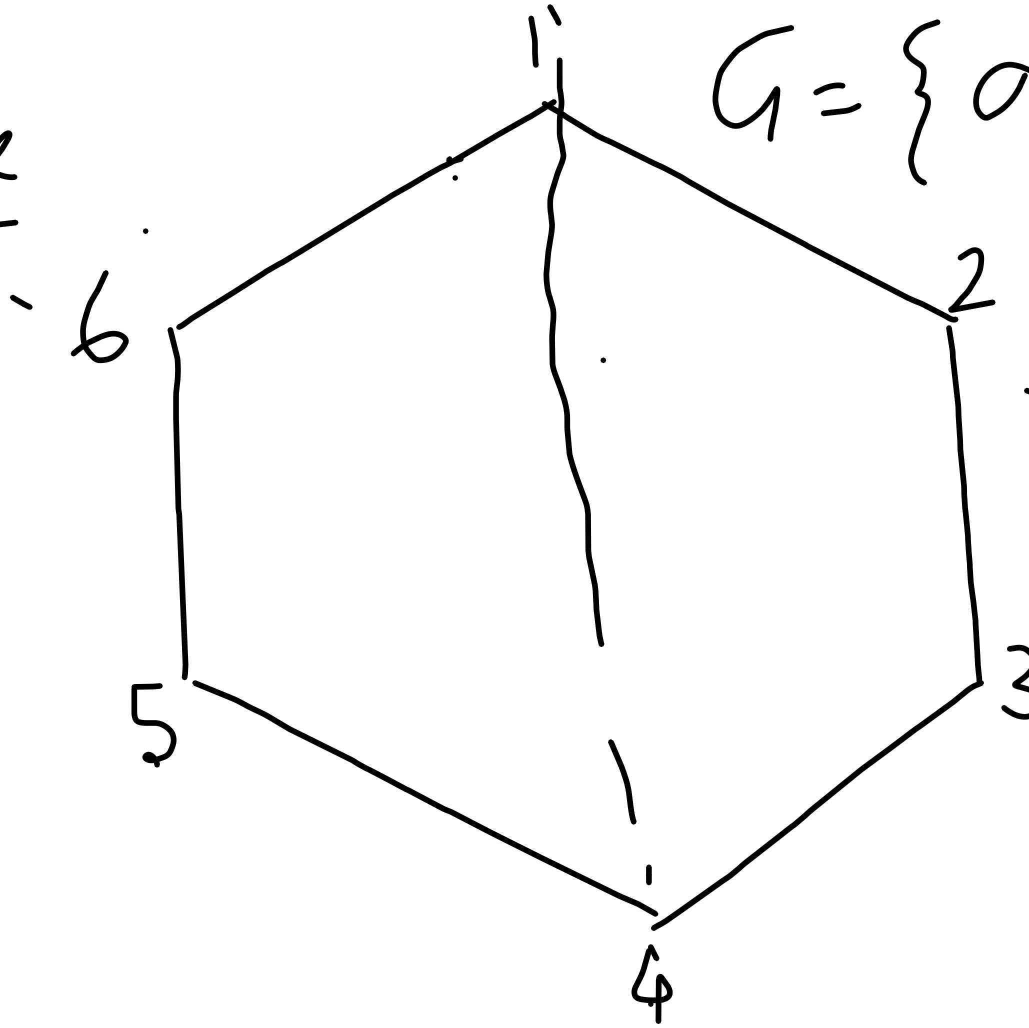
$\sigma(1) = n$

$\sigma(2) = (n-1)$

$\sigma^2 = i$

$\mathrm{Aut}(P_4) = \mathbb{Z}_2 \quad \mathbb{Z}/2\mathbb{Z}$

② Cycle



$G = \{\sigma, r, \sigma^6 = 1, r^2 = 1, r\sigma = \sigma^{-1} r\}$

Dihedral group

$r\sigma = \sigma^{-1} r$

$\sigma = \begin{pmatrix} 1 \to 2 \\ 2 \to 3 \\ \vdots \\ 5 \to 6 \\ 6 \to 1 \end{pmatrix}$

$i,$

$\begin{pmatrix} 1 \to 3 \\ 2 \to 4 \\ 3 \to 5 \\ 4 \to 6 \\ 5 - 1 \\ 6 \to 2 \end{pmatrix}$

$1 - 6$
$2 - 5$

$i, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5,$

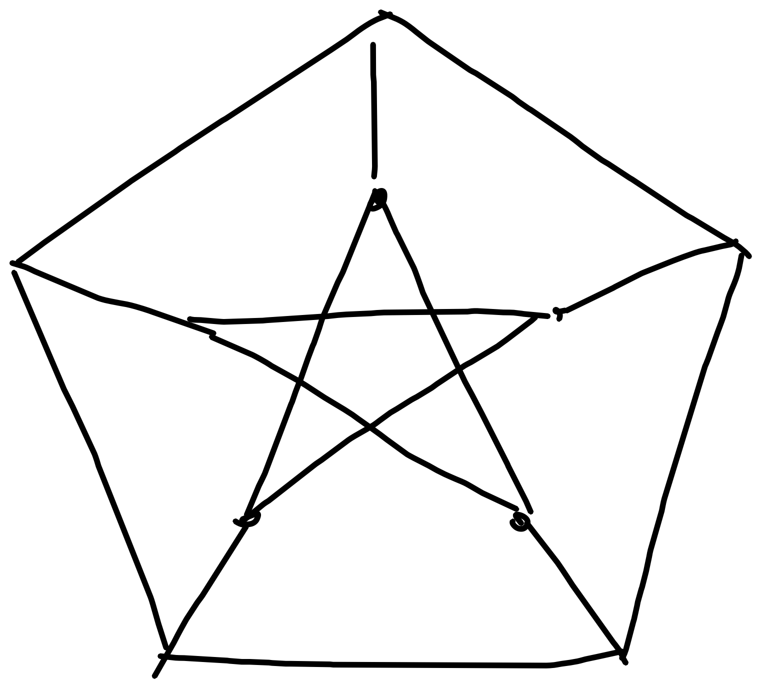
$r = \begin{pmatrix} 1 - 1 \\ 2 - 6 \\ 3 - 5 \\ 4 - 4 \end{pmatrix}$

$r^2 = \underline{i}$

3

Petersen graph

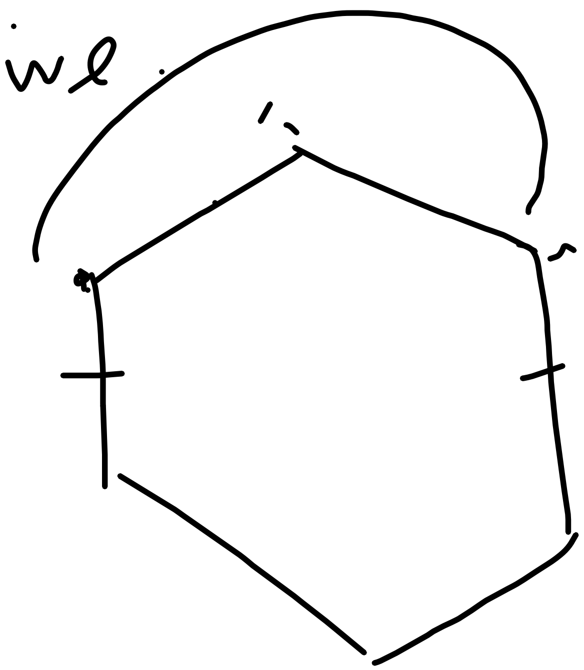
$\mathrm{Aut}(\mathrm{Pet})$



vertex transitive

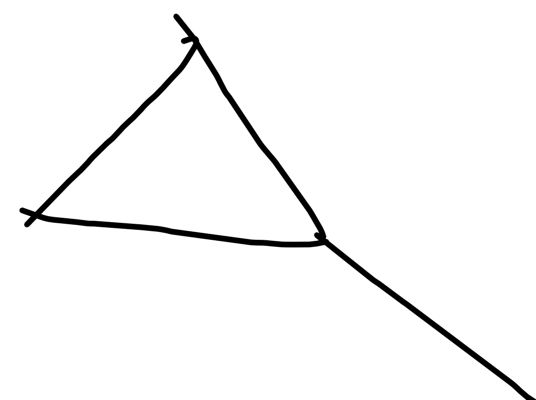
$v_1 \quad v_2$

$\sigma(v_1) = v_2$



Edge transitive

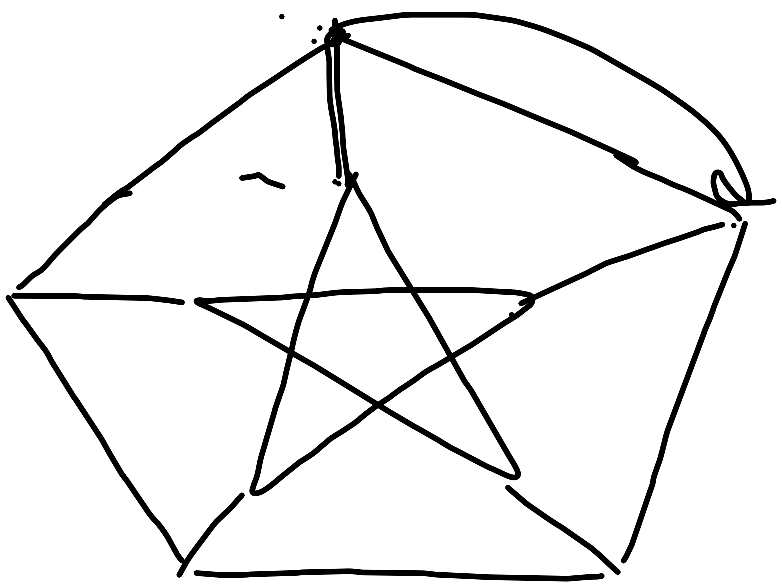
$(u_1, u_2) \qquad (v_1, v_2)$



$\sigma(u_1) = v_1$

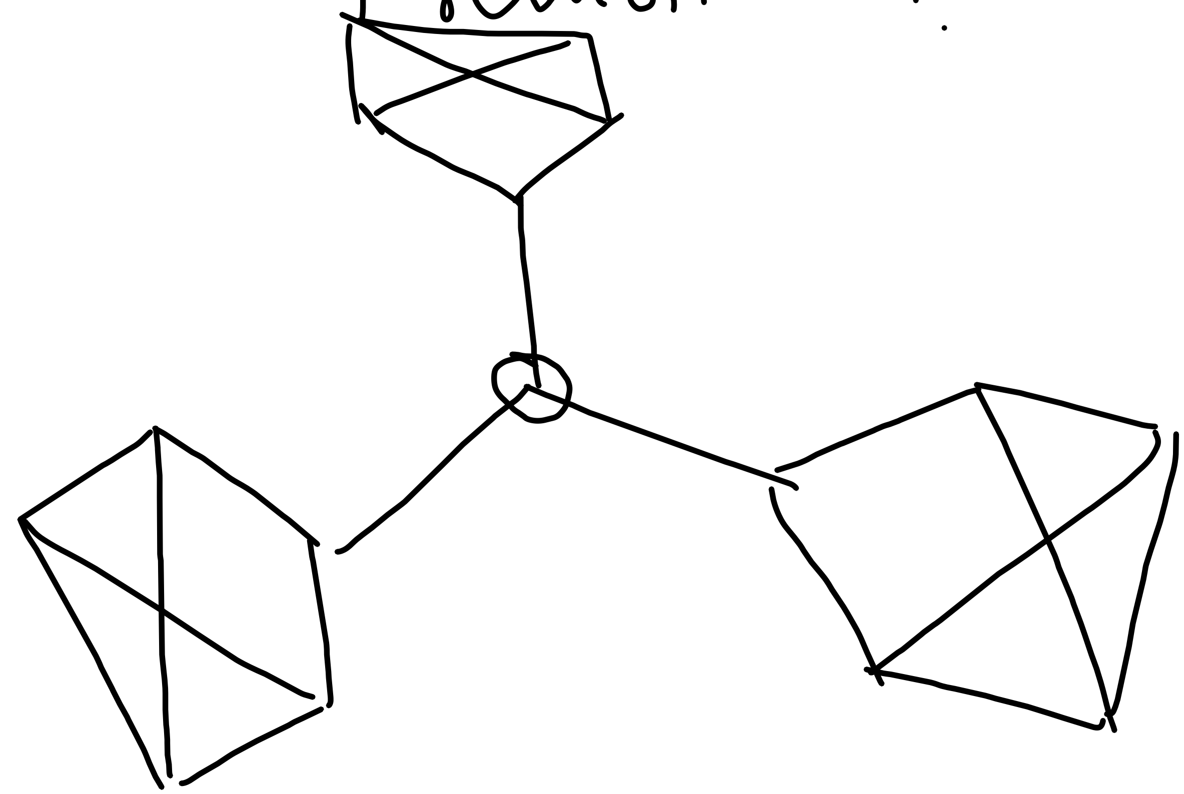
$\sigma(u_2) = v_2$

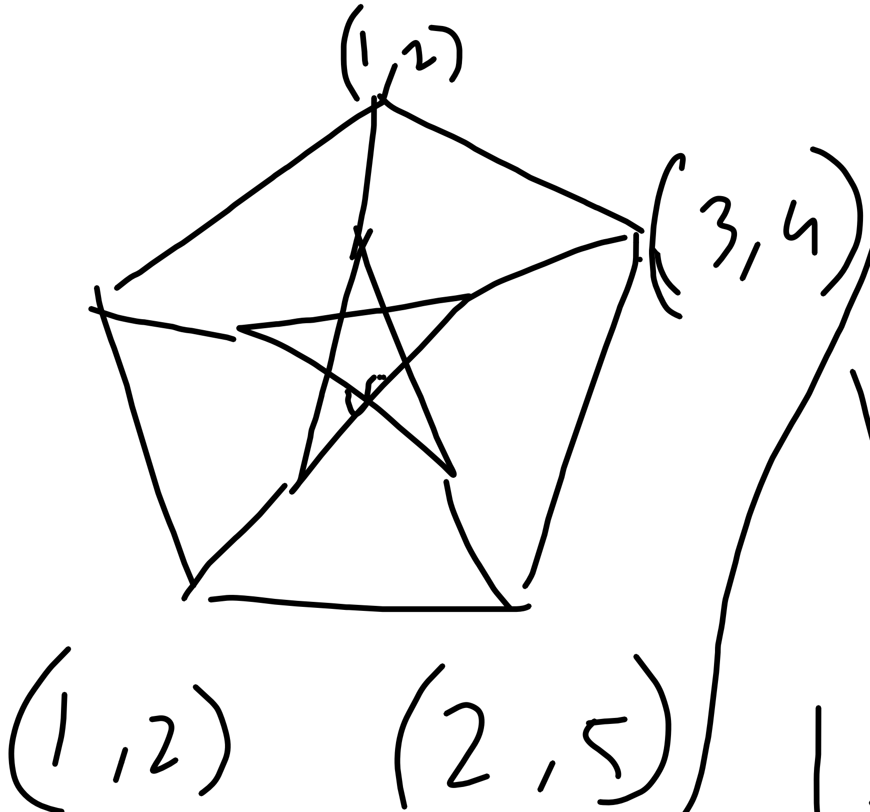
Aut(Pet).



$$\left|\operatorname{orbit}_{\operatorname{Aut}(\operatorname{Pet})}(v)\right| = \underline{10}$$

Does being regular mean that a graph is vertex transitive?





$n = \{1, 2, 3, 4, 5\}$

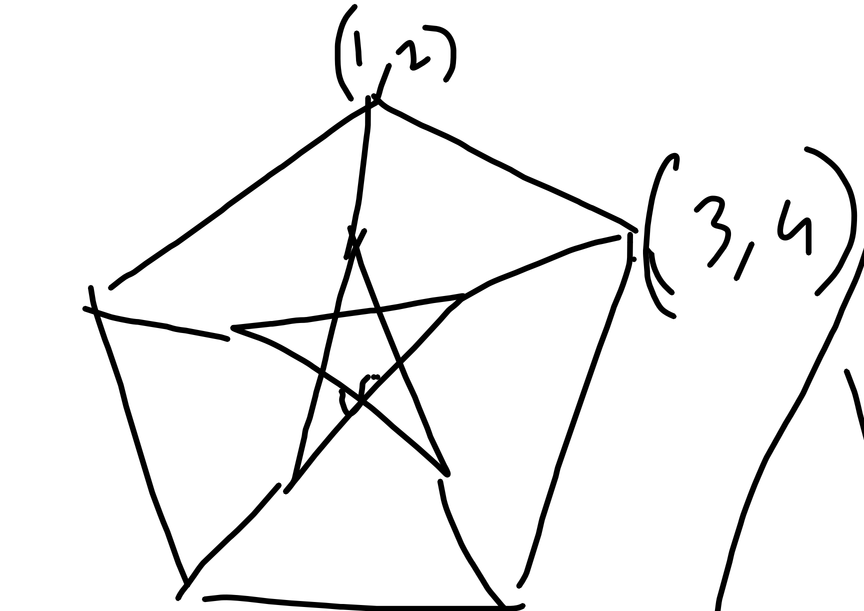
$V = \begin{bmatrix} n \\ 2 \end{bmatrix} = \{ (1,2) , (1\ 3) \cdots \cdots \}$

$|V| = 10$

$E_1 = (u_1, u_2) \sim E_2 = (v_1, v_2)$ if $E_1 \cap E_2 = \emptyset$

S_n

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$ Exc



(1,2) (2,5).

$n = \{1, 2, 3, 4, 5\}$

$V = \binom{n}{2} = \{ (1,2), (1\ 3) \ldots \ldots \}$

$|V| = 10.$

$E_1 = (u_1, u_2) \sim E_2 = (v_1, v_2)$, if $E_1 \cap E_2 = \emptyset$

S_n.

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$ Exc

$\mathrm{Aut}(\mathrm{Pet}) \leq \underline{S_n}$. $|S_n| = 5! = \underline{120}$

$|\mathrm{Orbit}(v)| = \underline{10}$.

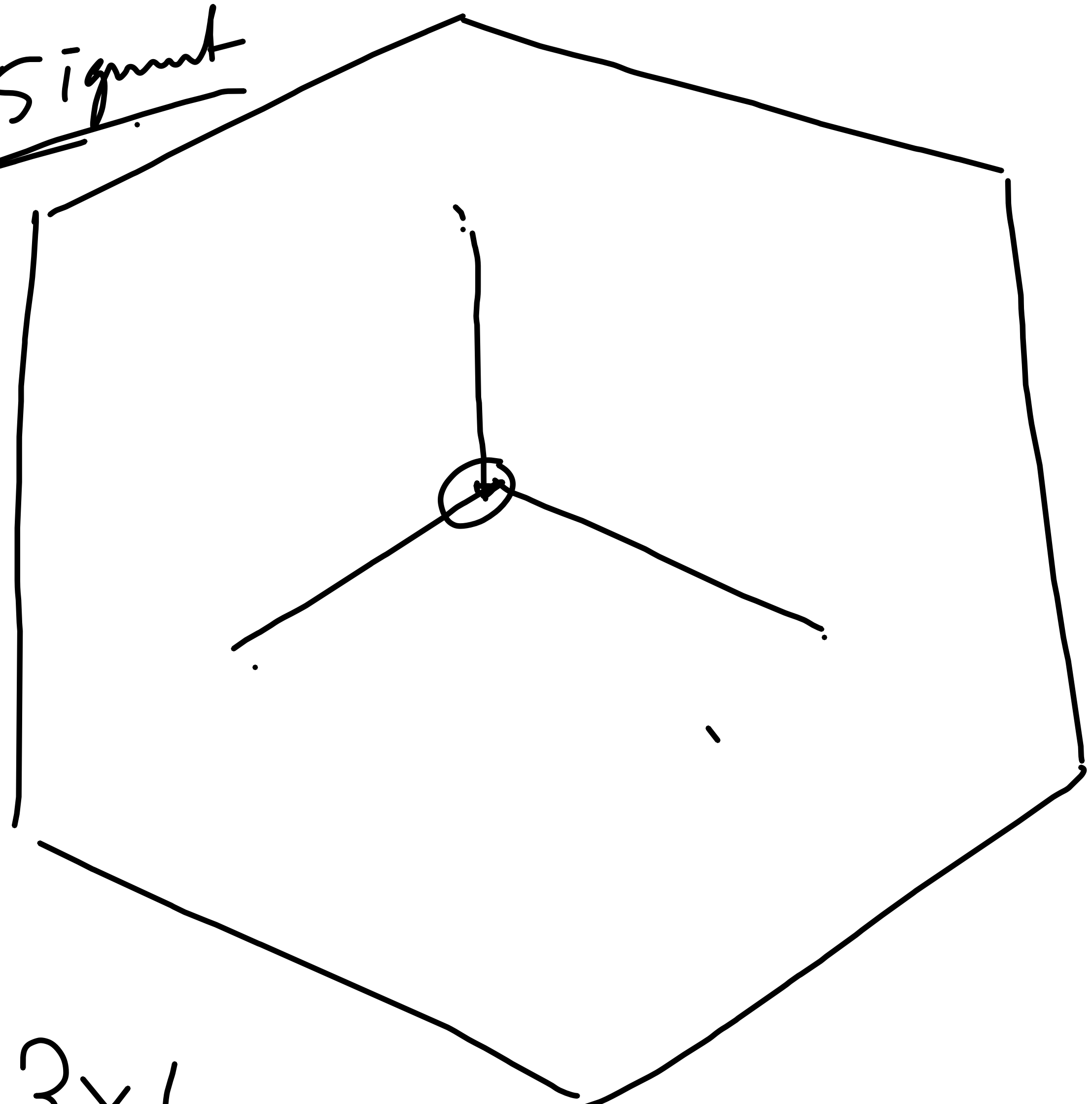
$|\mathrm{Orbit}(v)| \times |\mathrm{Stabilizer}(v)| = |\mathrm{Aut}(\mathrm{Pet})|$

$\underline{10}$. $\times$ 12. ?

$= \underline{120}$

$\overline{\mathrm{Aut}(\mathrm{Pet}) \approx \underline{S_5}}$.

Assignment



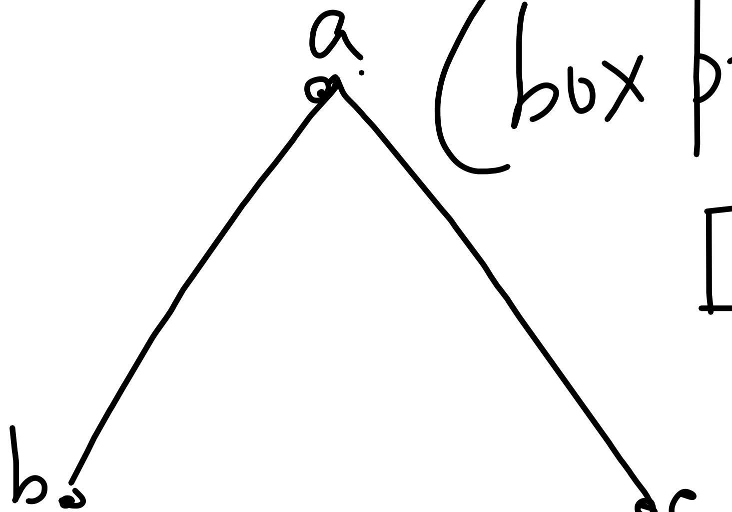
$|\mathrm{Stab}_v| = 12.$

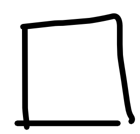
3×4

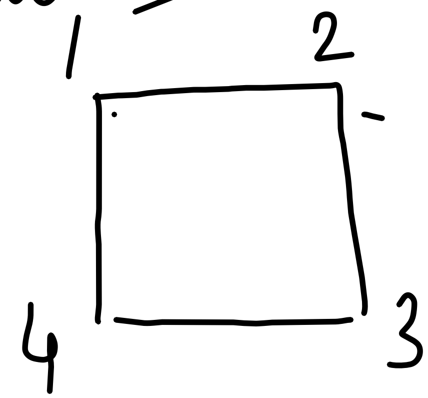
Graph products — $G(U)$, $H(V)$

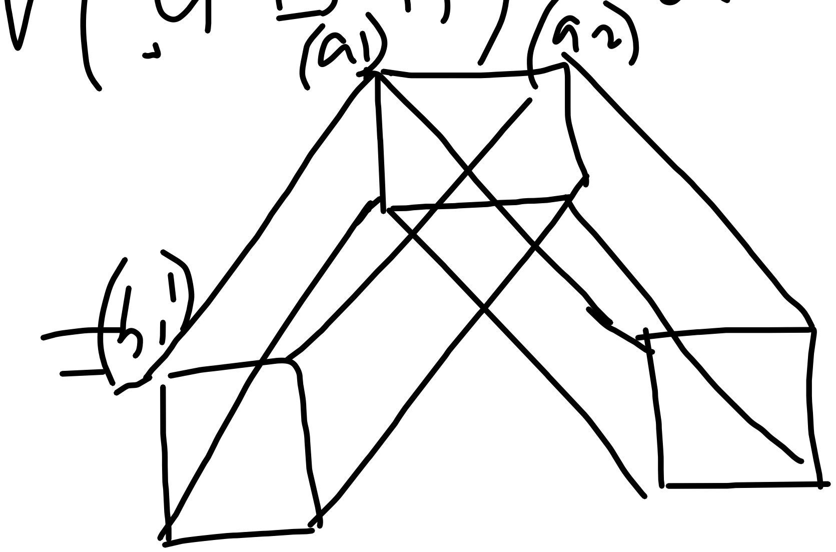
① Cartesian Product $V(G \square H) = U \times V$

(box product)









$(u, u') \sim (v, v')$ if ① $u = v$ and $u' \sim v'$

or

② $u' = v'$ and $u \sim v$

$(a,1), (a\,2), (a\,3), (a\,4)$

$(b1)$ $b2$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{bmatrix}$$

$$\left[\begin{matrix} 1 \times \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{bmatrix} & 2 \times \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{bmatrix} \\ 3 \times \begin{bmatrix} \; \end{bmatrix} & 4 \times \begin{bmatrix} \; \end{bmatrix} \end{matrix}\right]_{6 \times 6}$$

$$A_{1 \square 2} = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2$$

Tensor product

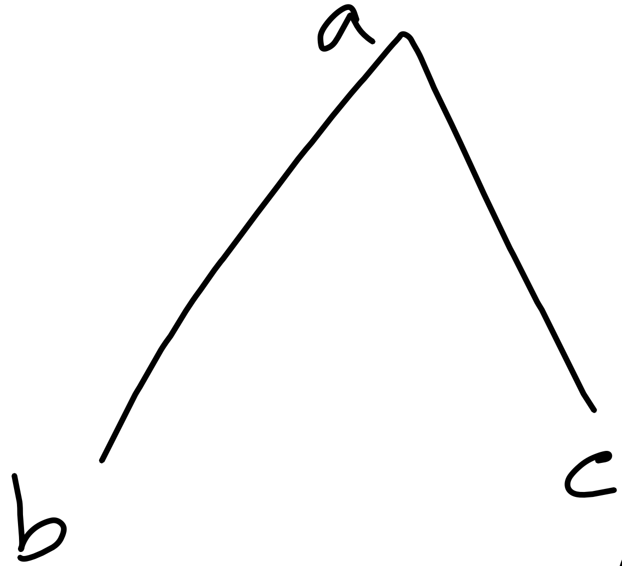
$G \otimes H$

$V = V(G) \times V(H)$

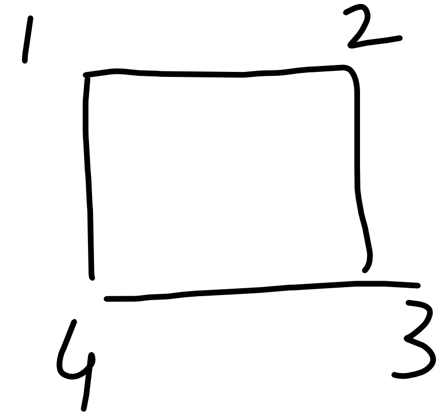
$A_{1 \otimes 2} = A_1 \otimes A_2$

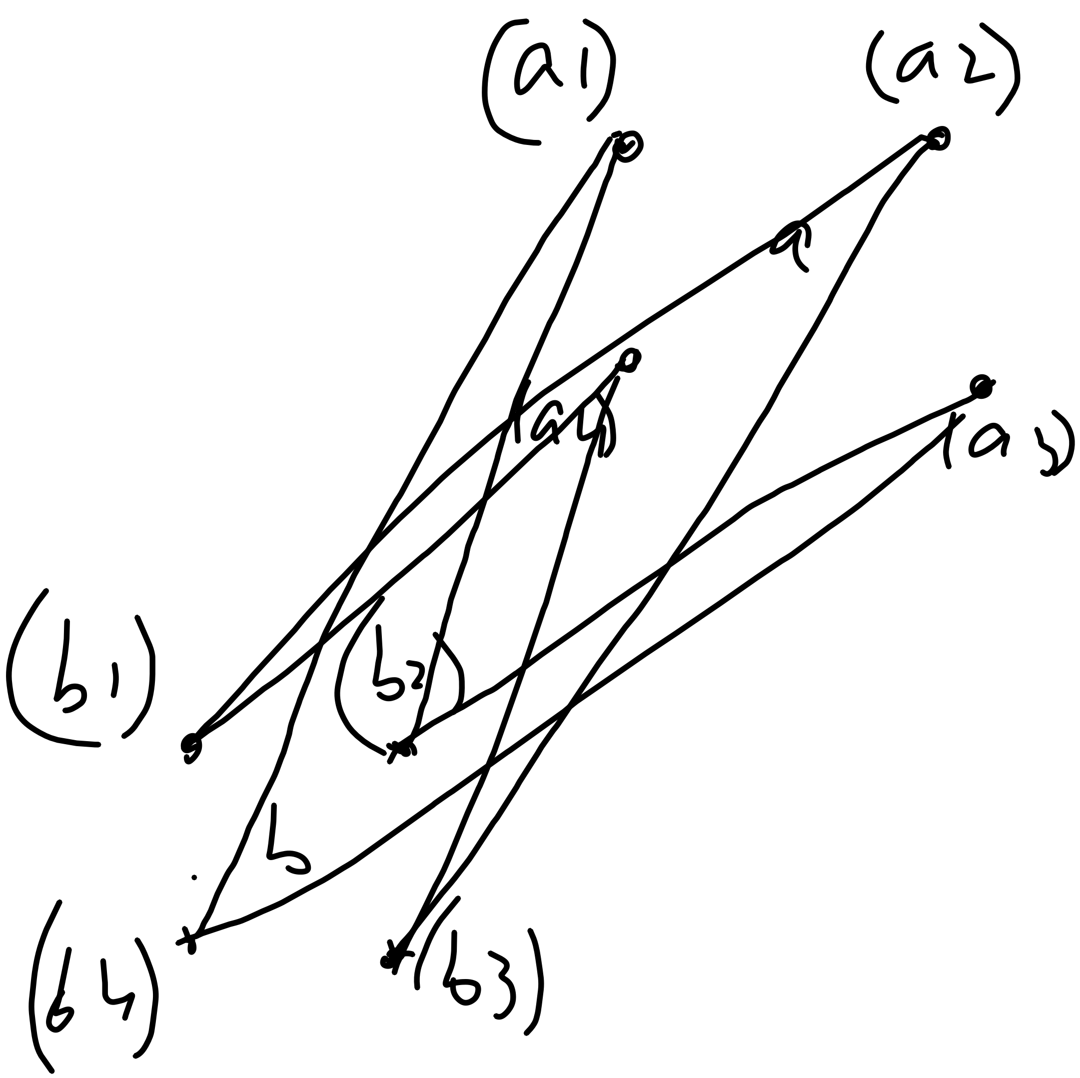
$(g, h) \sim (g', h')$

$g \sim g'$ in G and $h \sim h'$ in H



$\otimes$

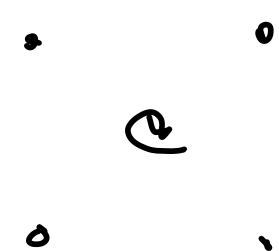




$(g, h), (g', h')$

$g \sim g'$

$h \sim h'$

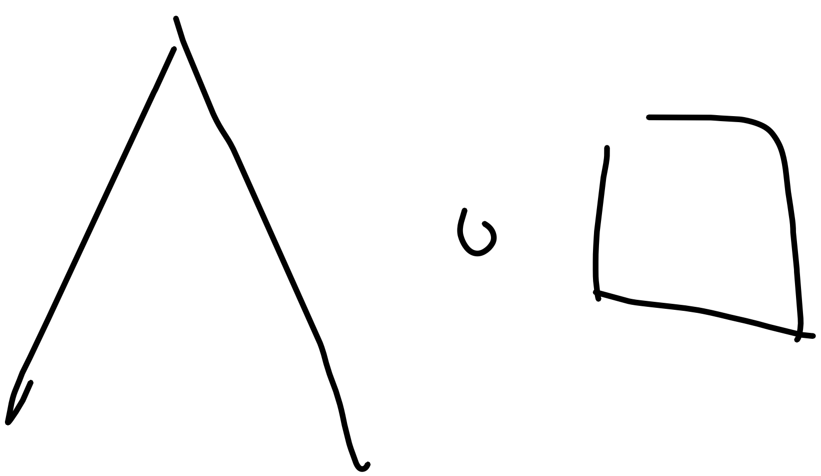


Lexicographic product: $G \circ H$.

$(g, h) \sim (g', h')$

either $g \sim g'$

or $g = g'$ and $h \sim h'$.



Strong product $G \times H$.

$$(g, h) \sim (g', h')$$

if ① $g = g'$ and $h \sim h'$

or ② $h = h'$ and $g \sim g'$

or ③ $g \sim g'$ and $h \sim h'$.

Wreath product $G \wr H$

$G = (V_G, E_G)$, $H = (V_H, E_H)$

$(K_3 \wr K_2 \neq K_2 \wr K_3)$

$$V(G \wr H) = V_H^{V_G} \times V_G$$

$$= \{ (f, \underline{v}) \mid f : V_G \to V_H,\ v \in V_G \}$$

$(f, v) \sim (f', v')$

① $v = v'$ and $f(w) = f'(w)$ $\forall w \neq v'$ and $f(v') \sim f'(v')$

or ② $f(w) = f'(w)$, $\forall w \in V(G)$ and $v \sim \underline{v'}$