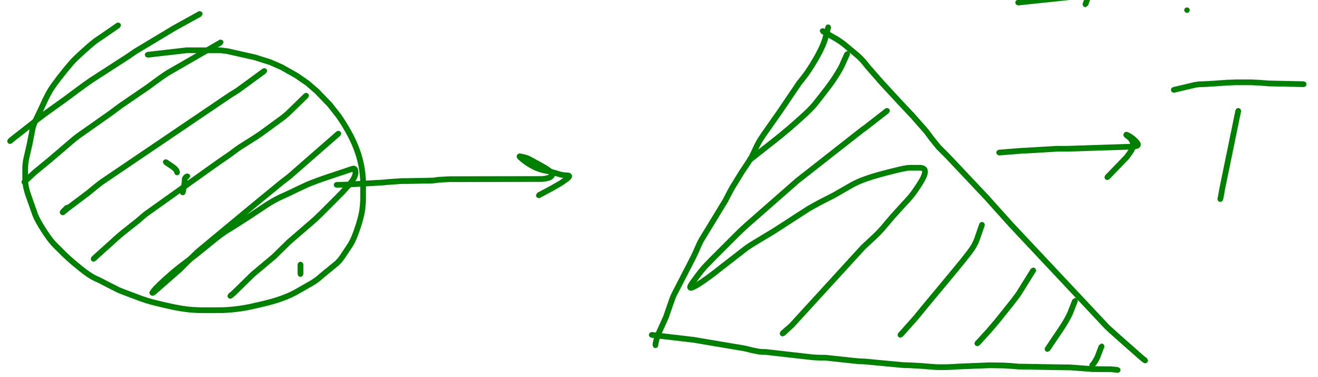
Brouwer's fixed pt theorem:-

If $f : D^2 \longrightarrow D^2$ is a continuous map the f has a fixed pt. $D^2 =$ unit circle with all the interior pts.

$f(x) = x$ —.



$f : T \longrightarrow T$

$f : D^n \longrightarrow D^n$

$f(x) - x \neq 0$

$$g(x) = \frac{f(x) - x}{|f(x) - x|}.$$

Brouwer's fixed pt theorem :-

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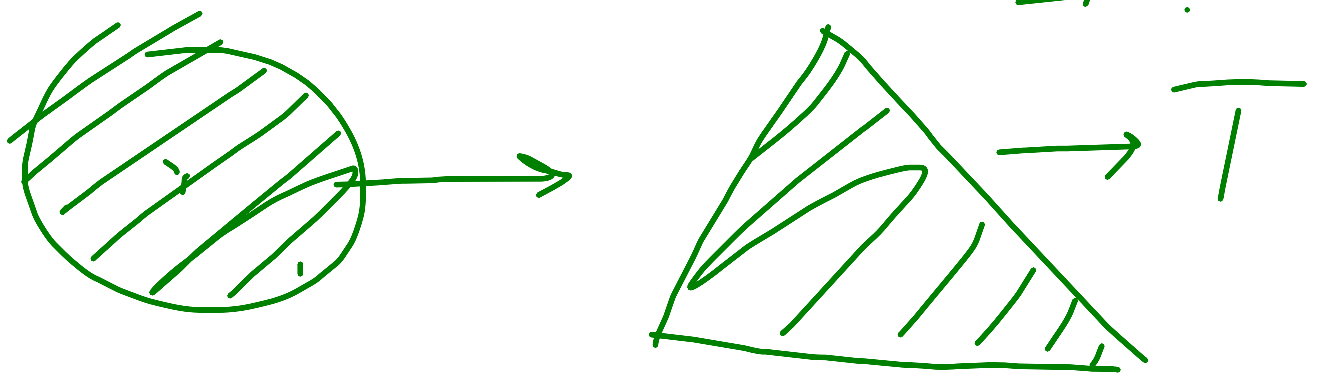
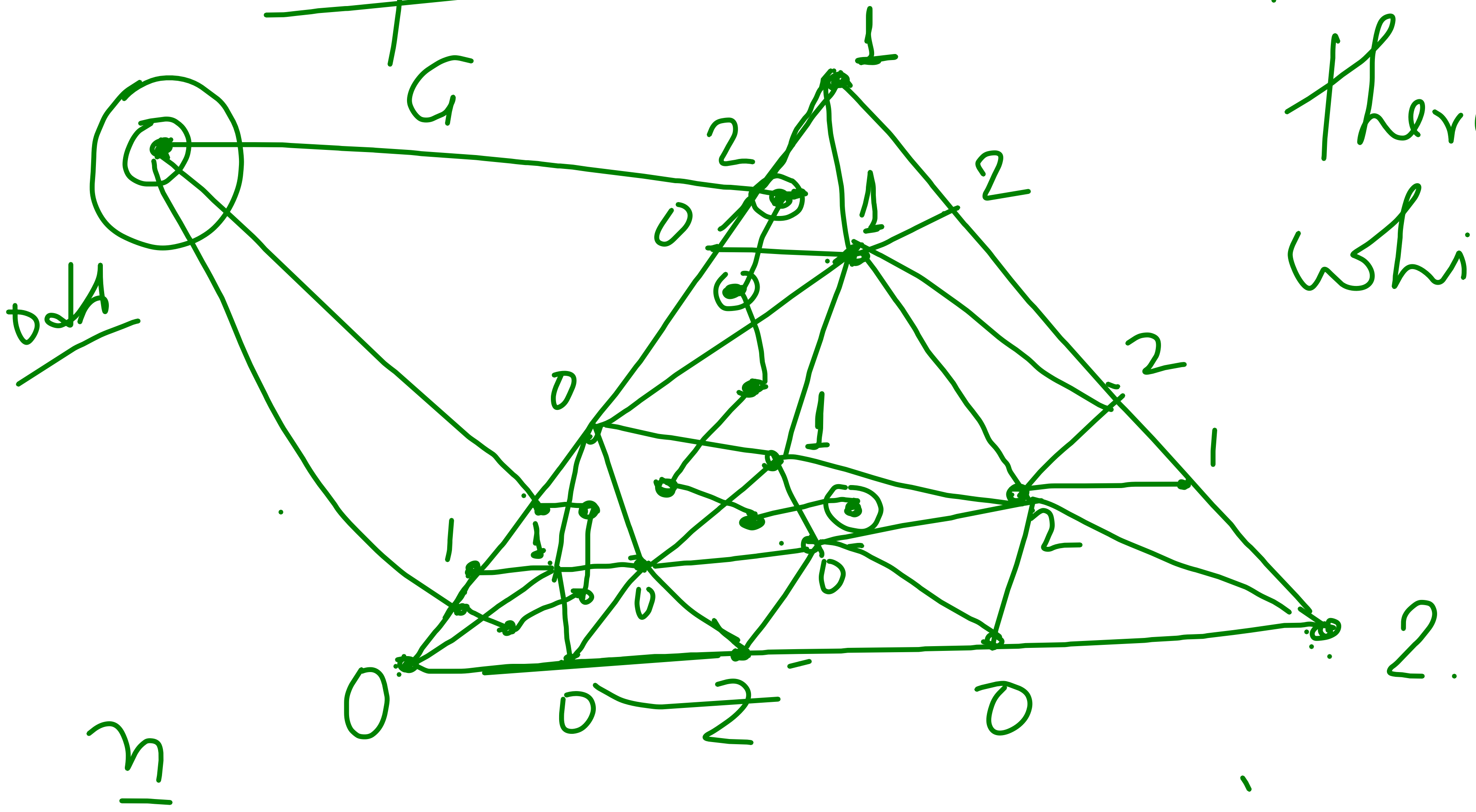


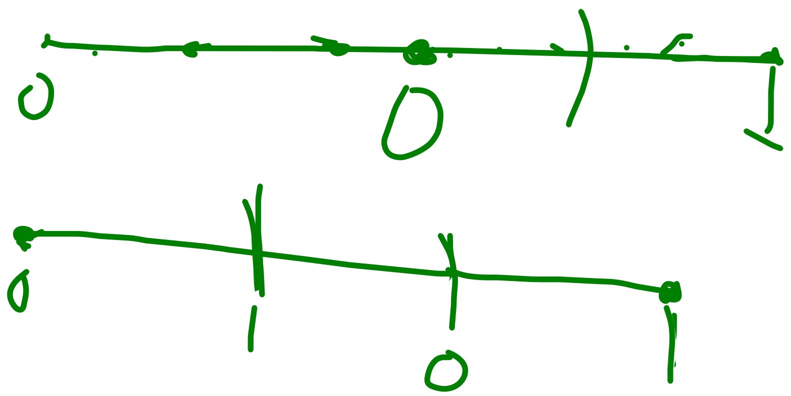
$f : T \longrightarrow T$

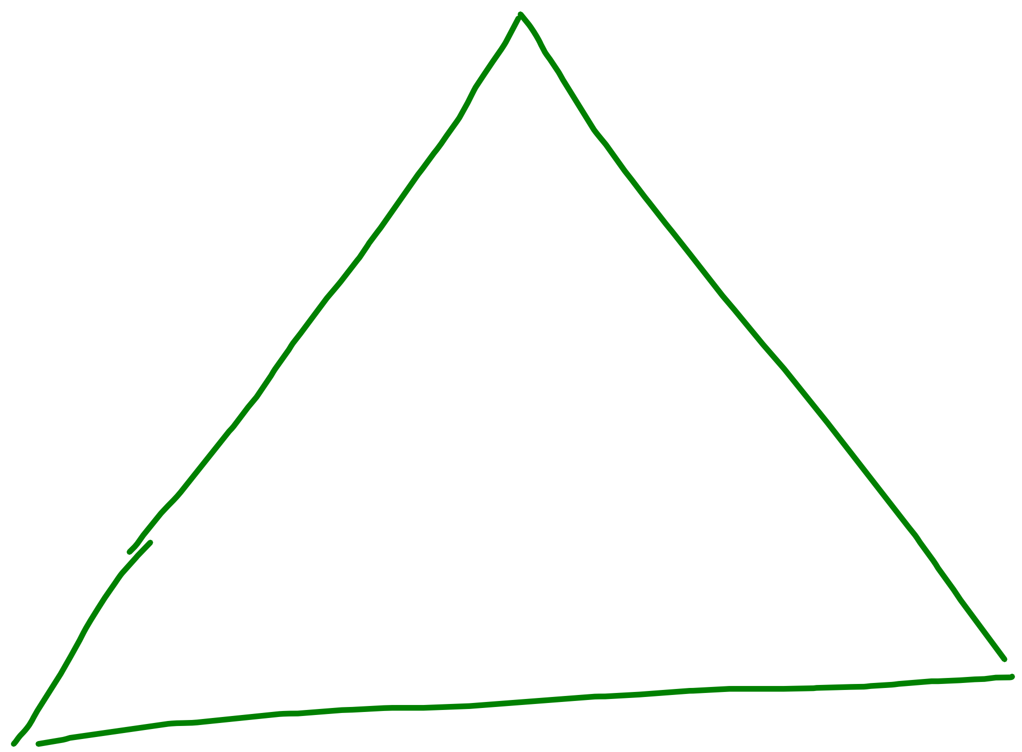
Sperner's lemma

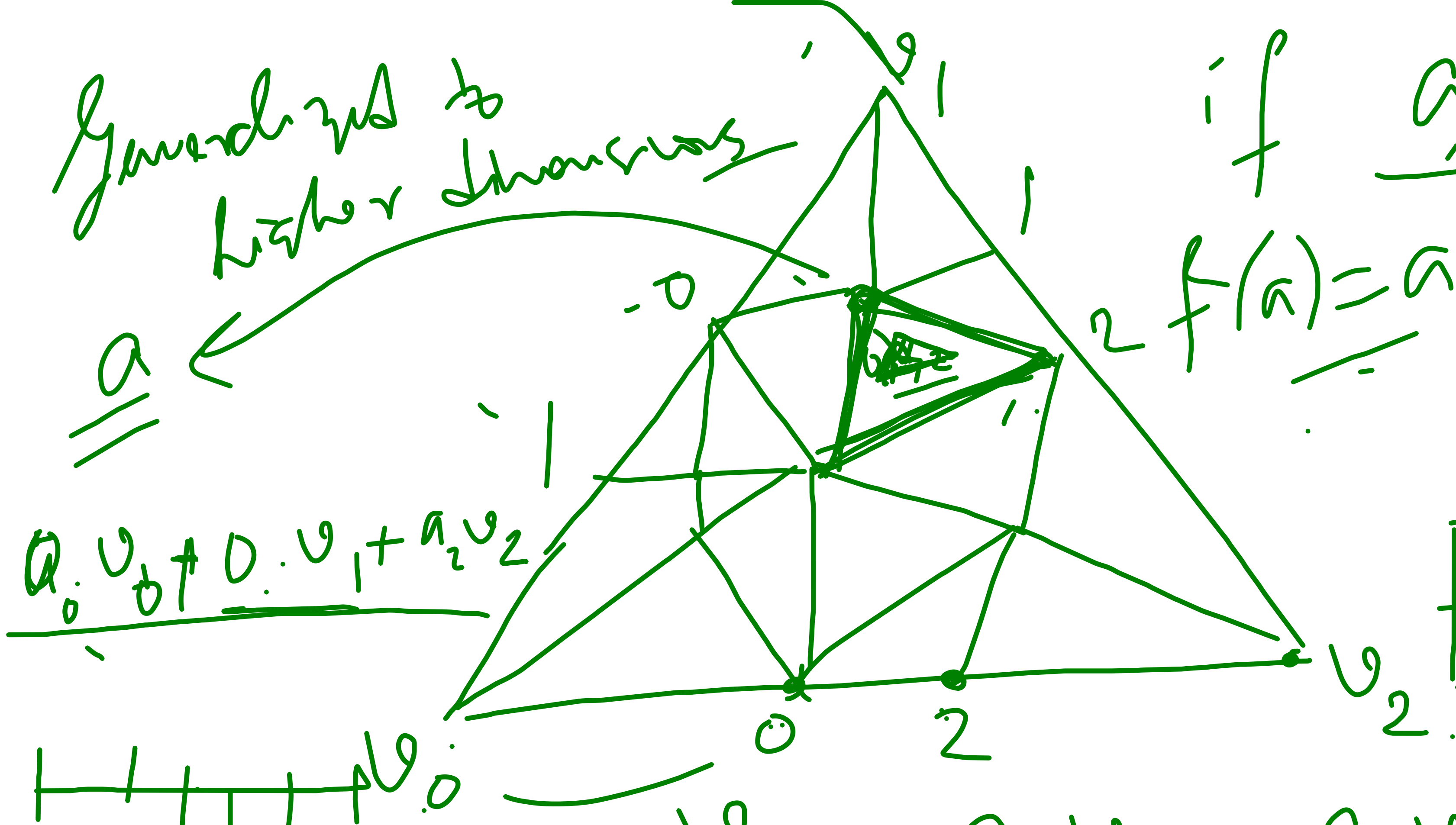


There must be a simplex which is coloured 0, 1, 2

Sperner colouring







$a_0 v_0 + 0 \cdot v_1 + a_2 v_2$

In: if $f: D^n \to D^n$

$v = a_0 v_0 + a_1 v_1 + a_2 v_2$

S_0, S_1, S_2

$\sum a_i = 1 \qquad a_i \geq 0$

$a \in S_0$ if $a_0' \leq a_0$

$a \in S_1$ if $a_1' \leq a_1$

$a \in S_2$ if $a_2' \leq a_2$

if $\underline{a} \in S_0, S_1, S_2$

$f(a) = a$:

$f(a_0 v_0 + a_1 v_1 + a_2 v_2)$

$= a_0' v_0 + a_1' v_1 + a_2' v_2$

$f(a_0, a_1, a_2)$

$\underline{a} = a_0' + a_1' + a_2'$

$a_0 \quad a_1 \quad a_2$

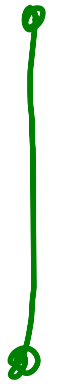
$\sum a_0' + a_1' + a_2' = 1$

$a_0 + a_1 + a_2 = 1$

Exe → Complete the proof in higher dimensions.

Colouring. A graph is k-chromatic if $\chi(G) = k$.
If $\chi(H) < \chi(G)$ for every proper subgraph H of G, then G is called color-critical or k-critical.

Example



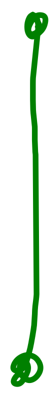
$\chi(G) = 2$

3-critical graph? $\longrightarrow$ odd cycle.

Clique number :- max size of a set of pairwise of a graph G adjacent vertices (clique) in G.

G . K_m .. $\omega(G)$.

Example

 — $\chi(G)=2$

3-critical graph? $\longrightarrow$ odd cycle.

Clique number of a graph G :- max size of a set of pairwise adjacent vertices (clique) in G.

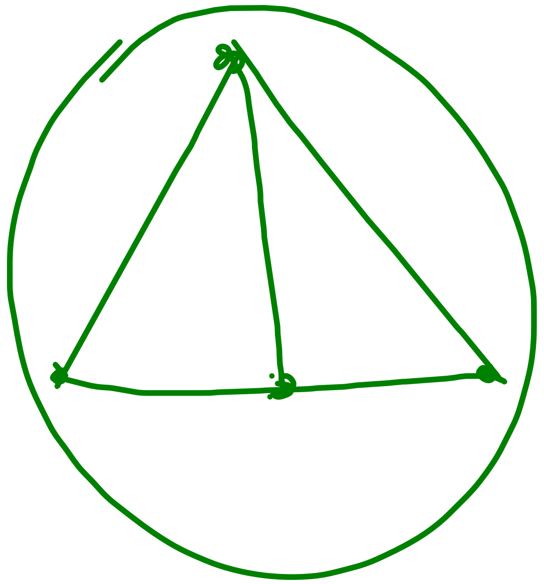
G . K_m .. $\omega(G)$.

Independence number :- maximum size of a set non-adjacent vertices in G. $\alpha(G)$

$\chi(G) \geq \omega(G)$.

tight for complete

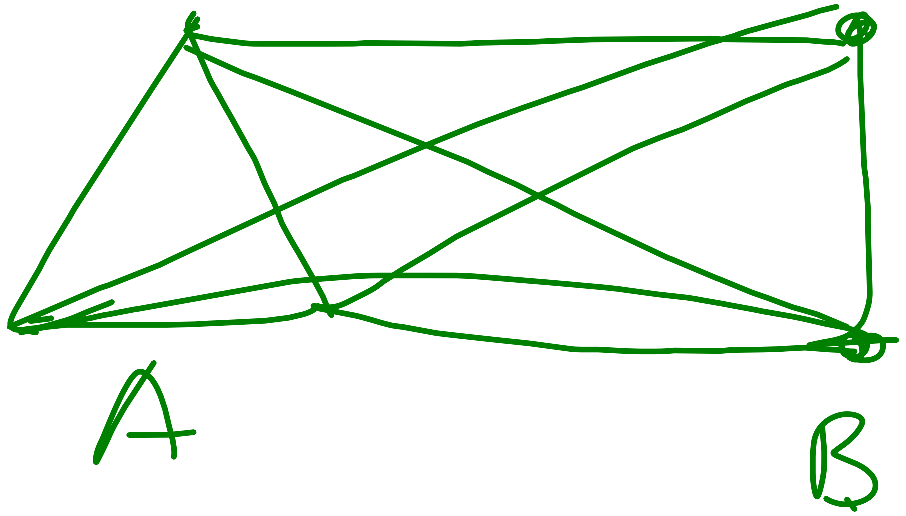
$\chi(G) > \omega(G)$?

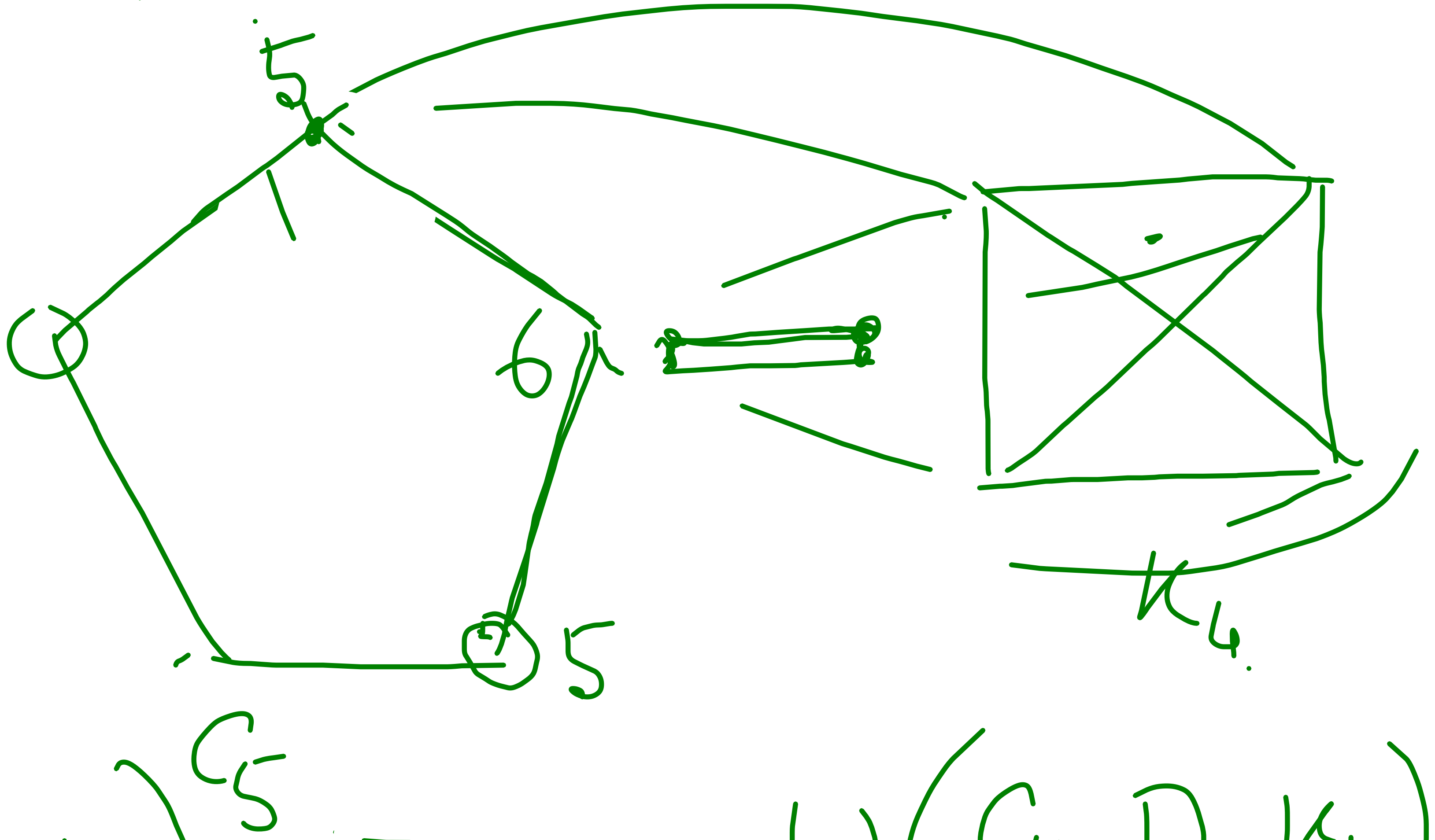


$\alpha(G)$. $n(G)$.

$$\chi(G) \geq \frac{n(G)}{\alpha(G)}$$

join of 2 graphs ?





$$\chi(G) = 7 > \omega(G) = 6$$

$$\chi(G) > \omega(G)$$

$$\chi(C_5 \square K_4) \geq 7.$$

$$\omega(C_5 \square K_4) = 4 + 2 = 6$$

K_6

Cartesian product G, H;

$G \,\square\, H$ $\quad \underline{V} = V(G) \times V(H)$.

$(u,v) \sim (u',v')$

iff ① $u = u'$ in G and $v \sim v'$ in H.

or

② $v = v'$ in H and $u \sim u'$ in G.

$X(G), X(H)$

Cartesian product G, H;

$G \,\square\, H$ $\quad \underline{V} = V(G) \times V(H)$.

$(u, v) \sim (u', v')$

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or

② $v = v'$ in H and $u \sim u'$ in G.

$\chi(G), \chi(H)$

Cartesian product G, H;

$G \square H$ $\quad \underline{V} = V(G) \times V(H)$.

$(u,v) \sim (u',v')$

iff ① $u = u'$ in G and $v \sim v'$ in H.

or

② $v = v'$ in H and $u \sim u'$ in G.

$X(G), X(H)$

$$\chi(G \square H) = \qquad \chi(G), \ \chi(H).$$

$$\underline{\chi(G \square H) = \max\{\chi(G), \chi(H)\}}$$

$$\geq \max\{\chi(G), \chi(H)\}$$

$$\leq k = \max\{\underline{\chi(G)}, \chi(H)\}$$

$$h : V(G) \longrightarrow [\chi(G)] = \{1, 2, \dots, \chi(G)\}$$

$$g : V(H) \longrightarrow [\chi(H)] = \{1, 2, \dots, \chi(H)\}$$

$$f(u, v) = (h(u) + g(v)) \bmod \underline{k}$$

$$f(u,v) = \big(h(u) + g(v)\big) \bmod (k)$$

$[0], [2], \ldots [k-1]$

$$(u,v) \sim (u', v')$$

$u = u'$, $v \sim v'$

$0, \ldots X(A)-1$

$$f(u,v) = \big(h(u) + g(v)\big) \bmod (k)$$

$$f(u',v') = \big(h(u') + g(v')\big) \bmod (k)$$

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$[\underline{0}], [\underline{2}], \dots [\underline{k-1}]$

$$(u,v) \sim (u',v')$$

$u = u'$, $v \sim v'$

$0, \dots X(k)-1$

$$f(u,v) = \big(h(u) + g(v)\big) \bmod (k)$$

$$f(u',v') = \big(h(u') + g(v')\big) \bmod (k)$$

How to colour a graph.

$$\chi(G) \leq n(G) \qquad \Delta(G) = \text{max degree.}$$

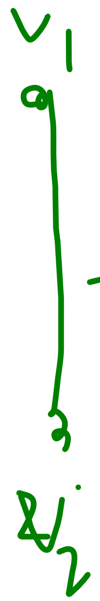
$$\chi(G) \leq \Delta(G) + 1 \ ?$$

Greedy colouring: G, $v_1, v_2, v_3, \ldots, v_n$.

assign to v_i to smallest indexed colour not already used on its lower indexed neighbours.

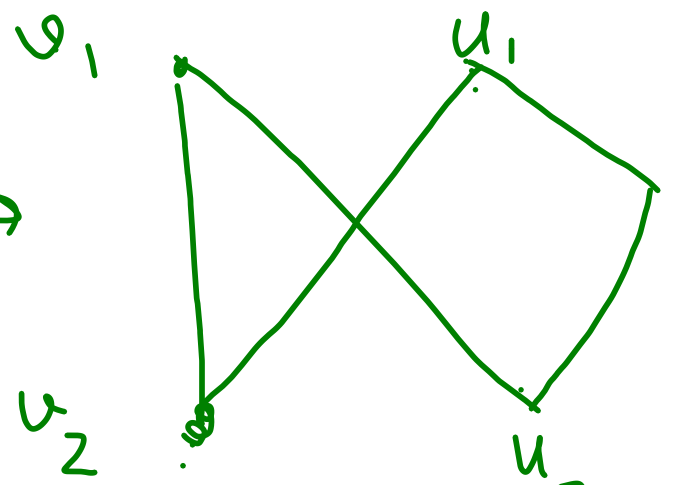
How to get a k-chromatic graph?

Mycielski's construction



2

$\mapsto$



3

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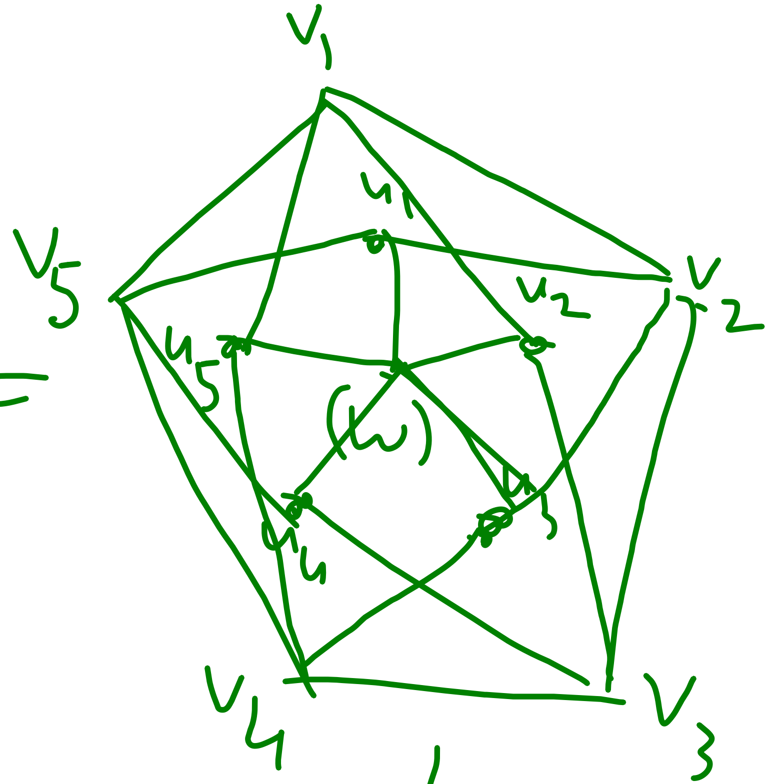


4

$5 \longrightarrow 6 \longrightarrow \cdots$

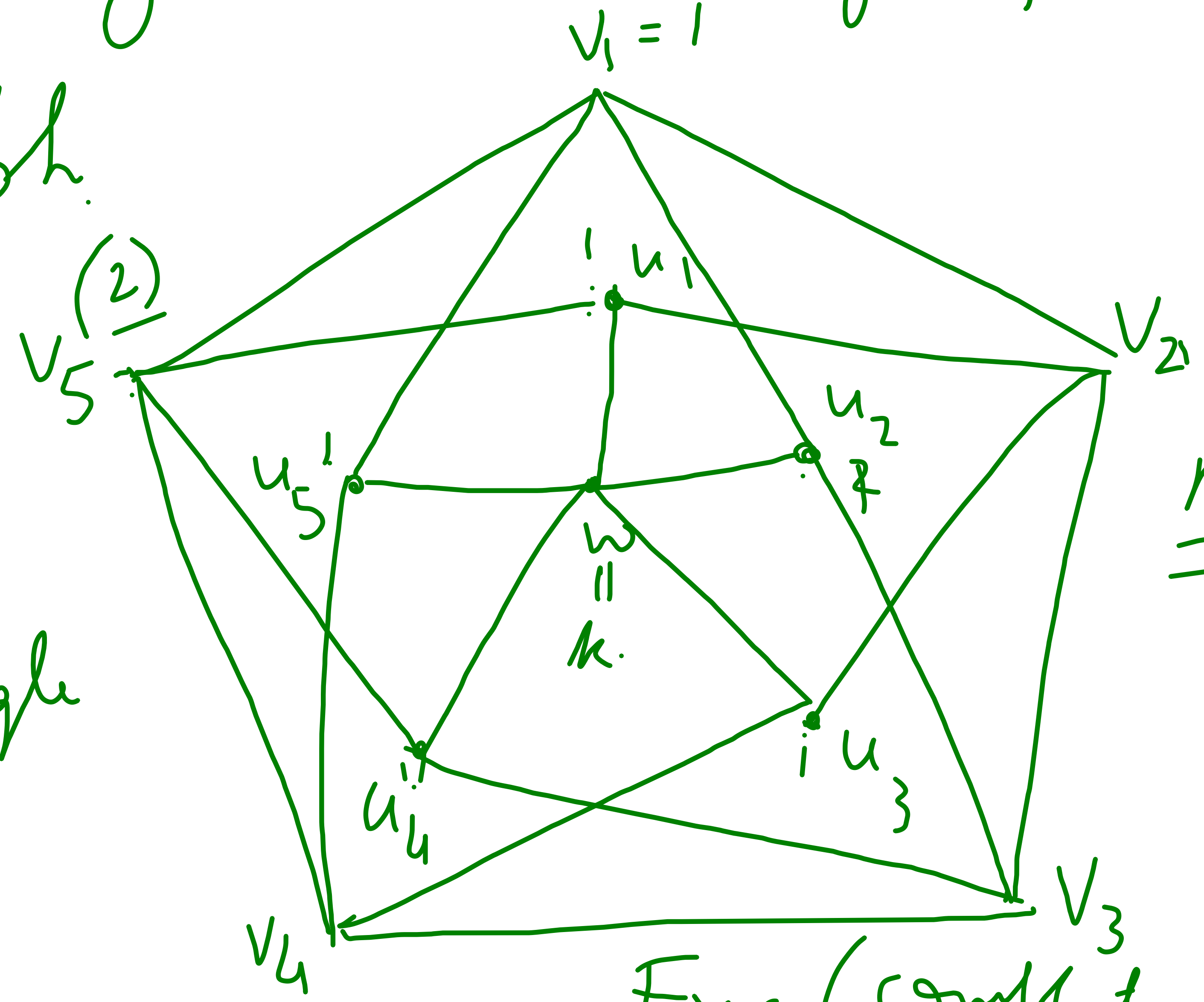
Starting with a triangle-free k-chromatic graph, M.C will give a triangle-free k+1 chromatic graph.

Proof :- G, $\chi(G) = k$

G is triangle-free.

$G \xrightarrow{M.C} G'$ is triangle free

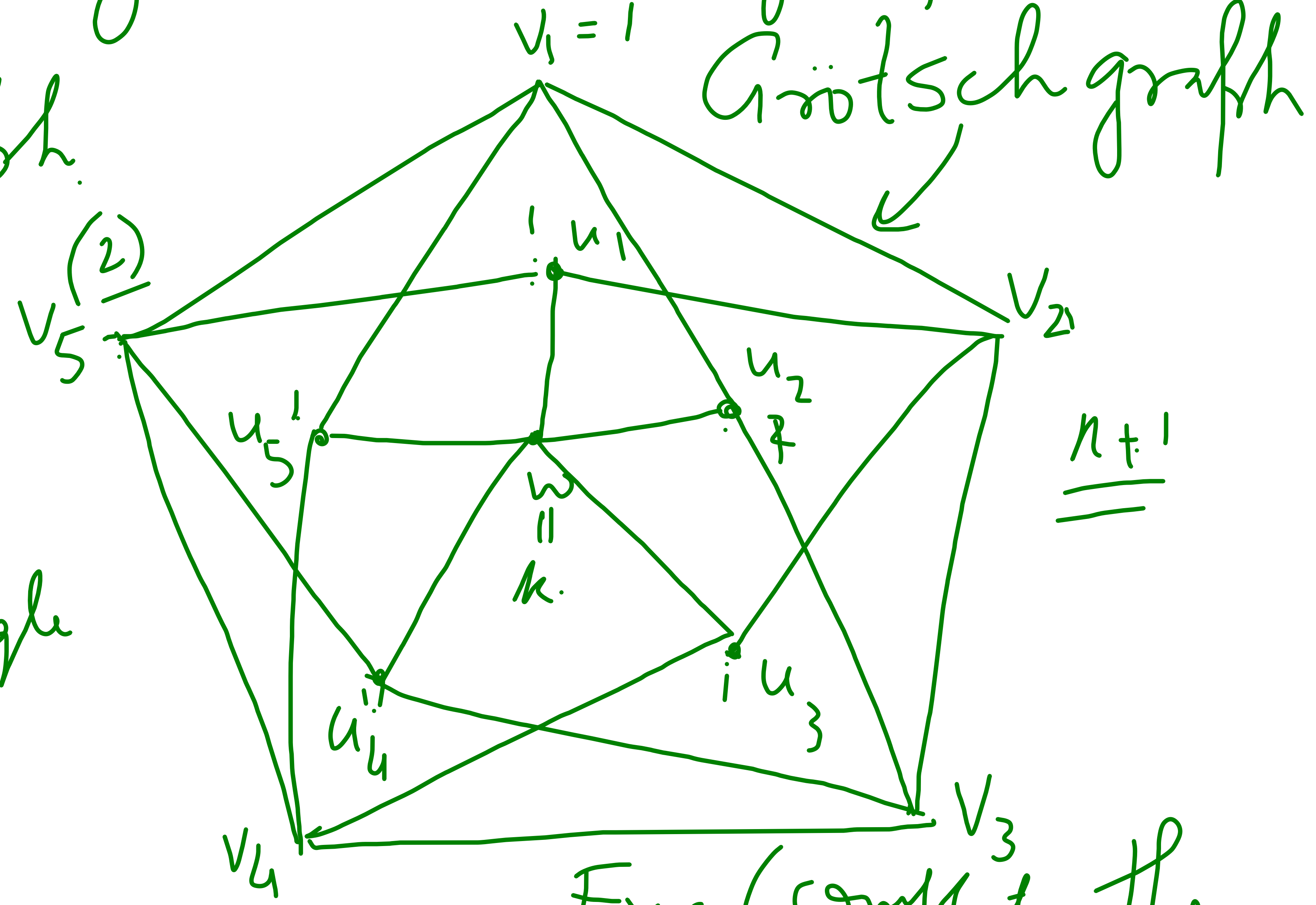
G' using k-colours



$\underline{k+1}$

Exc (complete the proof)

Starting with a triangle-free k-chromatic graph, M.C will give a triangle-free $k+1$ chromatic graph.



Proof :- G, $\chi(G) = k$

G is triangle-free.

$G \xrightarrow{M.C} G'$ is triangle free

G' using k-colours

$\chi(Pet)$? →

Exc (complete the proof)

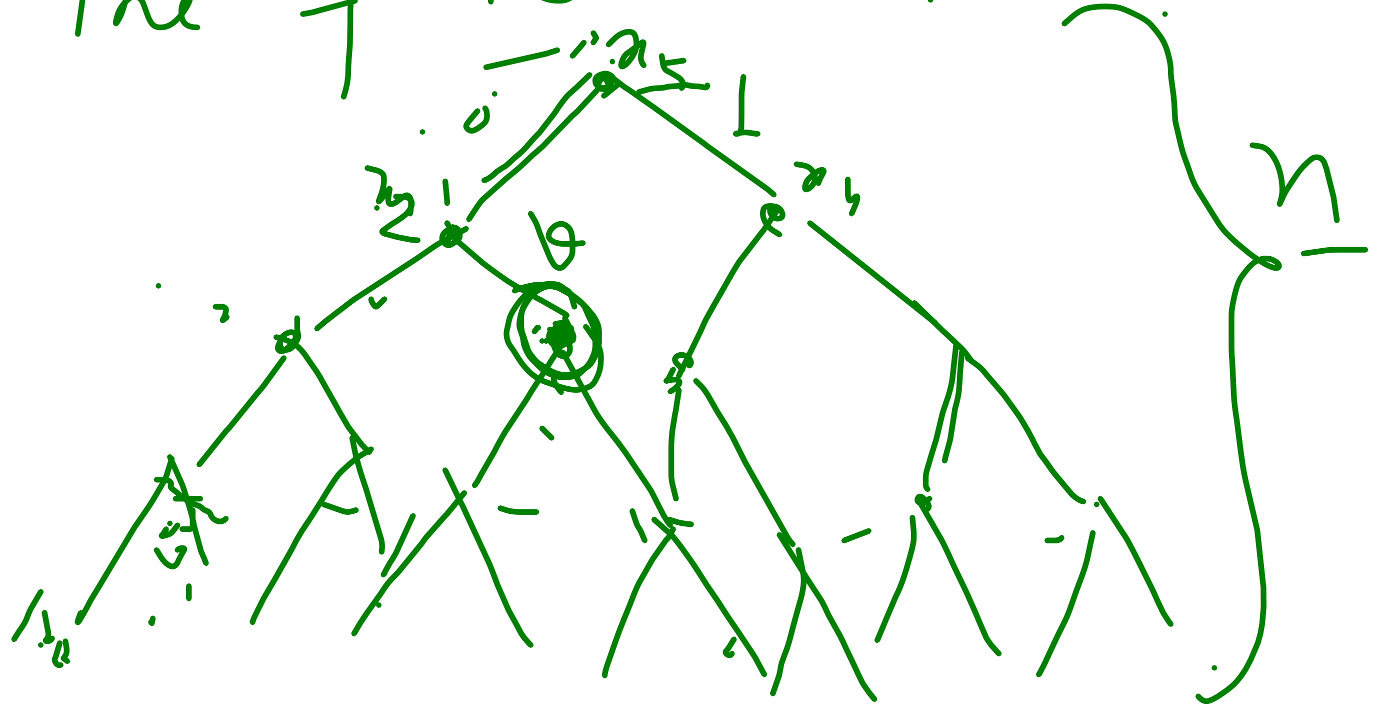
Exe. the graph obtained after M.C. is colour-critical

f is an evasive fn. $f:\{0,1\}^n \longrightarrow \{0,1\}$.

① If $|f^{-1}(0)|$ is odd the f is evasive

$f(0\,1\,0\,1\,0\,1\,0\,0\,1) = ?$

v is at a depth d $\quad 2^{n-d}$.



(2) A boolean string is even if it contains an even no. of 1's, odd otherwise.

Claim: If f is not evasive the exactly half the strings in $f^{-1}(0)$ are even

$(\ , \ ) + 2^{k}$

Evasive