# Linear Programming (LP)

Maximize | Minimize the <u>objective</u> function

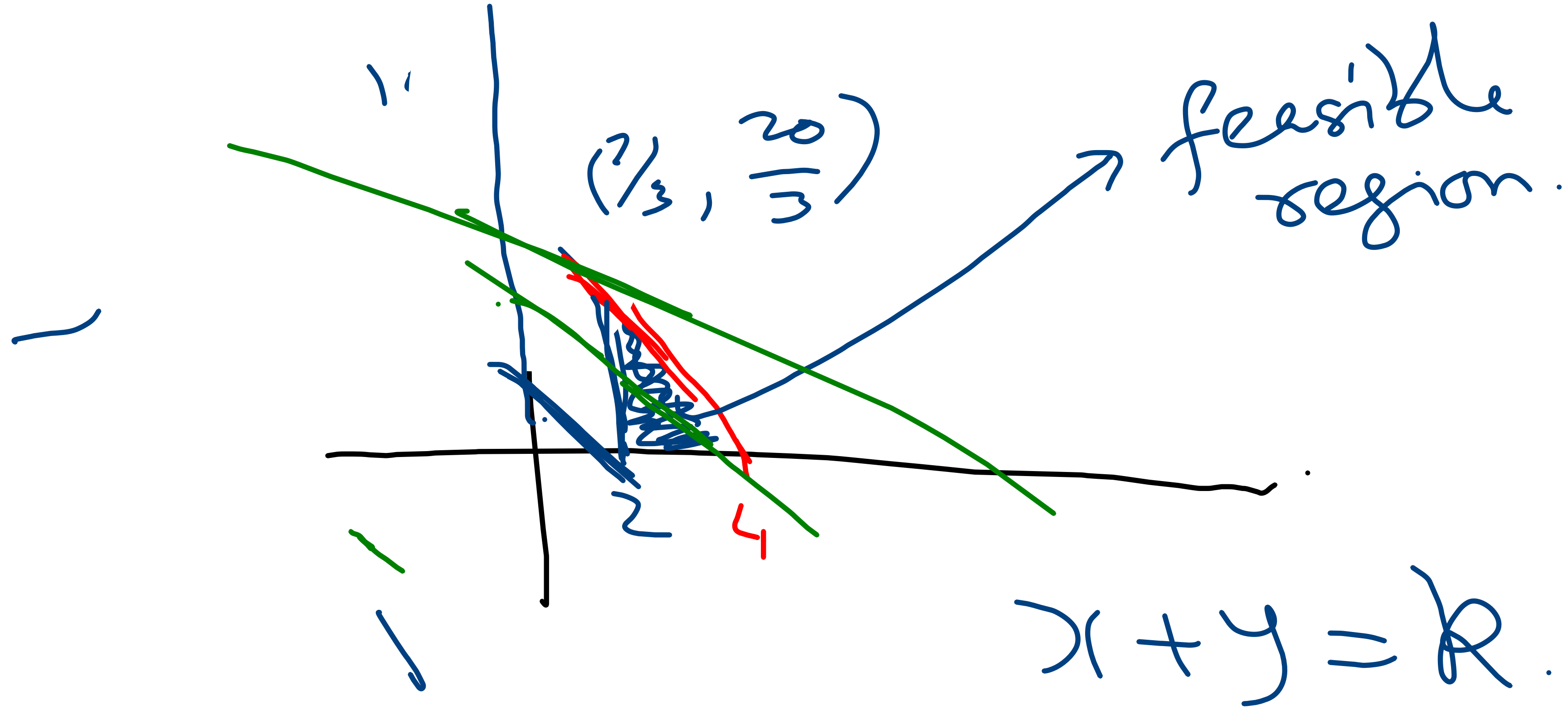
1. $Z = c_1 x_1 + c_2 x_2 + \cdots + c_r x_r$

Subject to the following $m$ constraints

2. $a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{ir} x_r \le \mid \ge \mid = b_i, \quad i = 1, \ldots, m$

3. $x_i \ge 0$.

## Example.




$x + y = k.$

Maximize

$x + y$

$$2x + y \le 8.$$

$$5x + y \ge 10$$

$$x, y \ge 0.$$

$$x = \frac{2}{3}$$

$$y = 8 - \frac{4}{3} = \frac{20}{3}$$

1. The feasible region is a convex set.
2. The optimum is attained at an extreme point.

Any vector $\underline{x} = (x_1, \ldots, x_n)$ satisfying (2) is called a <u>solⁿ</u>.

Any solⁿ that satisfy (3) is called a <u>feasible solⁿ</u>.

Any vector $\underset{\sim}{x} = (x_1, \ldots, x_n)$ satisfying (2) is called a <u>soln</u>.

Any soln that satisfy (3) is called a <u>feasible soln</u>.

# Slack and surplus variable

Consider an inequality constraint of the form

$$a_{h1}x_1 + a_{h2}x_2 + \cdots + a_{hr}x_r \leq b_h.$$

In this case we introduce a new variable $x_{r+h} \geq 0$ s.t.

$$a_{h1}x_1 + \cdots + a_{hr}x_r + x_{r+h} = b_h.$$

Such a variable is called a slack variable.
For a constraint of the form

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kr}x_r \geq b_k.$$

we introduce a new variable $x_{r+k} \geq 0$ s.t.

$$a_{k1}x_1 + \cdots + a_{kr}x_r - x_{r+k} = b_k.$$

Thus our LP problem reduces to the following

Find a vector $\underset{\sim}{x} = (x_1, \ldots, x_r) \geq 0$ satisfying ~~the~~ following system of eq$^{\underline{ns}}$

$$a_{h1} x_1 + a_{h2} x_2 + \cdots + a_{hr} x_r + x_{r+h} = b_h \quad , h = 1, \ldots, u$$

$$a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kr} x_r - x_{r+h} = b_k \quad , k = u+1, \ldots, v.$$

$$a_{p1} x_1 + \cdots + a_{pr} x_r = b_p \quad , p = v+1, \ldots, m$$

Find a vector $\underset{\sim}{x} = (x_1, \ldots, x_r) \geq 0$ satisfying the following system of eq$^{\underline{ns}}$

$$a_{h1} x_1 + a_{h2} x_2 + \cdots + a_{hr} x_r + x_{r+h} = b_h \quad , h = 1, \ldots, u$$

$$a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kr} x_r - x_{r+h} = b_k \quad , k = u+1, \ldots, v$$

$$a_{p1} x_1 + \cdots + a_{pr} x_r = b_p \quad , p = v+1, \ldots, m$$

This can be written in the matrix form as follow.

Find a vector $\underset{\sim}{x} = (x_1, \ldots, x_n) \geq 0$

satisfying $A\underset{\sim}{x}^T = \underset{\sim}{a}_1 x_1 + \underset{\sim}{a}_2 x_2 + \cdots + \underset{\sim}{a}_n x_n = \underset{\sim}{b}^T$ (4)

that maximizes | minimizes

(5) $z = \underset{\sim}{c} \cdot \underset{\sim}{x}^T$, where

$\underset{\sim}{a}_i$ are the columns of $A$, $r \leq n \leq r+m$

$\underset{\sim}{b} = (b_1, \ldots, b_m)$

$\underset{\sim}{c} = (c_1, \ldots, c_n)$, $c_i = 0$ if $i > r$.

Ex Consider the following constraints

$$x_1 + 7x_2 \le 10 \Rightarrow x_1 + 7x_2 + x_3 = 10.$$

$$2x_1 + 5x_2 \ge 4 \Rightarrow 2x_1 + 5x_2 - x_4 = 4.$$

$$x_1 + 3x_2 = 5 \Rightarrow x_1 + 3x_2 = 5$$

$$A = \begin{pmatrix} 1 & 7 & 1 & 0 \\ 2 & 5 & 0 & -1 \\ 1 & 3 & 0 & 0 \end{pmatrix}$$

Note that the matrix $A$ in (4) is of the form.

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1r} & 1 & 0 & \cdots & & 0 \\
a_{21} & a_{22} & \cdots & a_{2r} & 0 & 1 & 0 & \cdots & 0 \\
\cdots & & & & & & & 1 & \\
a_{u+1,1} & a_{u+1,2} & \cdots & a_{u+1,r} & & & & & -1 \\
& & & & & & & & & -1 \\
& & & & & & & & & & \ddots
\end{pmatrix}
$$

Claim Any feasible solⁿ that satisfies (4) and optimize (5) is equivalent to a feasible solⁿ of (2) that optimizes (1).

Suppose $\underline{x} = (x_1, \ldots, x_r)$ is a feasible solⁿ of (2)

Our method of introducing slack & surplus variables give rise to a feasible solⁿ $\underline{x}^* = (x_1, \ldots, x_n)$

<u>Claim</u> Any feasible sol$^n$ that satisfies (4) and optimize (5) is equivalent to a feasible sol$^n$ of (2) that optimizes (1).

Suppose $\underline{x} = (x_1, \ldots, x_r)$ is a feasible sol$^n$ of (2)
Our method of introducing slack & surplus variables give rise to a feasible sol$^n$ $\underline{x}^* = (x_1, \ldots, x_n)$

In $\underset{\sim}{x}$ optimizes (1), then $\underset{\sim}{x}^*$ will optimize (2)

<u>Lemma 1.</u> Consider $A\underset{\sim}{x}^T = \underset{\sim}{a}_1 x_1 + \cdots + \underset{\sim}{a}_n x_n = \underset{\sim}{b}^T$ where $r(A) = m$. Then for any feasible sol$^n$ $\underset{\sim}{x} \geq 0$, there is a <u>basic</u> feasible sol$^n$

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Since $r(A) = m$, $A$ has $m$ linearly independent col$^{ns}$. Form a matrix $B$ with these $m$ lin. ind. col$^{ns}$

If $\underset{\sim}{x}$ optimizes ①, then $\underset{\sim}{x}^*$ will optimize ②

<u>Lemma 1.</u> Consider $A\underset{\sim}{x}^T = \underset{\sim}{a}_1 x_1 + \cdots + \underset{\sim}{a}_n x_n = \underset{\sim}{b}^T$ where $r(A) = m$. Then for any feasible sol$^{\underline{n}}$ $\underset{\sim}{x} \geq 0$, there is a <u>basic</u> feasible sol$^{\underline{n}}$

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Since $r(A) = m$, $A$ has $m$ linearly independent col$^{\underline{ms}}$. Form a matrix $B$ with these $m$ lin. ind. col$^{\underline{ms}}$

Then any sol$^{n}$ of $B\hat{\underset{\sim}{x}}^T = \underset{\sim}{b}^T$ gives rise to a <u>basic</u> sol$^{n}$

$$\underset{\sim}{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m, 0, \ldots, 0)$$

Pf of Lemma 1

Consider a feasible sol$^n$ with $p$ +ve variable $x_j$, $j = 1, 2, \ldots, p$.

$x_j = 0$ for $j > p$.

Case 1 Suppose $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_p$ are lin. independent

Clearly, $p \leq m$.

If $p = m$ then we obtain a non-degenerate basic feasible sol$^n$

If $p < m$, set the remaining $m-p$ variables to be 0. & we obtain a degenerate basic feasible soln

<u>Case 2.</u> $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_p$ are dependent.

Then $\exists$ scalars $\alpha_1, \ldots, \alpha_p$ (not all zeros)

s.t. $\alpha_1 \underset{\sim}{a}_1 + \ldots + \alpha_p \underset{\sim}{a}_p = \underline{0}$ 6

Fix an $\alpha_r$ s.t $\alpha_r > 0$ (such an $\alpha_r$ exists)

From (6) we have

$$\underset{\sim}{a}_r = -\sum_{\substack{j=1 \\ j \neq r}}^{p} \frac{\alpha_j}{\alpha_r}\, \underset{\sim}{a}_j \qquad (7)$$

Note that we have

$$x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2 + \cdots + x_r \underset{\sim}{a}_r + \cdots + x_p \underset{\sim}{a}_p = \underset{\sim}{b}$$

$r^{th}$ term

$$\underbrace{\left(x_1 - \frac{\alpha_1}{\alpha_r} x_r\right)}_{\hat{x}_1} \underset{\sim}{a}_1 + \underbrace{\left(x_2 - \frac{\alpha_2}{\alpha_r} x_r\right)}_{\hat{x}_2} \underset{\sim}{a}_2 + \cdots 0 \cdots + \underbrace{\left(x_p - \frac{\alpha_p}{\alpha_r} x_r\right)}_{\hat{x}_p} \underset{\sim}{a}_p = \underset{\sim}{b}$$

Choose $\underset{\sim}{a}_r$ s.t.

$$\frac{x_r}{\alpha_r} = \min\left\{ \frac{x_j}{\alpha_j} : \alpha_j > 0 \right\}$$

Set $\hat{x}_j = x_j - \frac{\alpha_j}{\alpha_r} x_r$.

So we obtain a basic feasible sol$^n$ with at most $p-1$ +ve variables.

Proceed as above until we obtain a basic feasible sol$^n$ as in case 1.