$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}, \quad \text{where } x, y \geq 0$$

$$\& \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 0.$$

$$f(x, y) = x^{\frac{1}{p}} y^{\frac{1}{q}}, \quad x, y \geq 0.$$

We need to maximize $f(x, y)$ subject to $\frac{x}{p} + \frac{y}{q} = \sigma$, where σ is an arb. const.

$$\mathcal{L} = f(x, y) - \lambda\left(\frac{x}{p} + \frac{y}{q} - \sigma\right)$$

$$\mathcal{L}_x = \frac{1}{p} x^{\frac{1}{p}-1} y^{\frac{1}{q}} - \frac{\lambda}{p}$$

$$\mathcal{L}_y = \frac{1}{q} x^{\frac{1}{p}} y^{\frac{1}{q}-1} - \frac{\lambda}{q}.$$

At an extremum, we have

$$\frac{1}{p} x^{\frac{1}{p}-1} y^{\frac{1}{q}} = \frac{\lambda}{p} \Rightarrow \frac{1}{\cancel{p}x} f(x,y) = \frac{\lambda}{\cancel{p}} \Rightarrow x = \frac{f(x,y)}{\lambda}.$$

$$\frac{1}{q} x^{\frac{1}{p}} y^{\frac{1}{q}-1} - \frac{\lambda}{q} = 0 \Rightarrow \frac{1}{\cancel{q}y} f(x,y) = \frac{\lambda}{\cancel{q}} \Rightarrow y = \frac{f(x,y)}{\lambda}.$$

$$\frac{1}{p} f(x,y) + \frac{1}{q} f(x,y) = \frac{\lambda x}{p} + \frac{\lambda y}{q} = \lambda \sigma$$

$$\Rightarrow f(x,y) = \lambda \sigma.$$

Let (x_0, y_0) be the pt. where f is maximized.

We have

$$\frac{f(x_0, y_0)}{\lambda} = \sigma.$$

Hence x_0, y_0 satisfy.

$$x_0 = \frac{f(x_0, y_0)}{\lambda} = \sigma$$

$$y_0 = \frac{f(x_0, y_0)}{\lambda} = \sigma$$

Hence we have $\forall\, x, y \geq 0$.

$$f(x,y) = x^{\frac{1}{p}} y^{\frac{1}{q}} \leq f(x_0, y_0) = f(\sigma, \sigma) = \sigma$$

$$= \frac{x}{p} + \frac{y}{q}$$

Find max or min of the quadratic

form $Q(\underset{\sim}{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$, where

$\underset{\sim}{x} = (x_1, \ldots, x_n)$ subject to the

constraint $\sum_{i=1}^{n} x_i^2 = \underline{1}$

Find max or min of the quadratic form $Q(\underset{\sim}{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$, where

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$Q(\underset{\sim}{x}) = \underset{\sim}{x} \cdot A \cdot \underset{\sim}{x}^T$, where $A = (a_{ij})$ &

A is symmetric.

Let

$$L = Q(\underset{\sim}{x}) - \lambda \sum_{i=1}^{n} x_i^2.$$

$$L_{x_i} = \sum_{j=1}^{n} a_{ij} x_j - 2\lambda x_i$$

$$\cancel{2} \cdot \sum_{j=1}^{n} a_{ij} x_j = \cancel{2}\lambda x_i \quad , \quad i = 1, \ldots, n.$$

$$A_{(i)} \cdot \underset{\sim}{x} = \lambda x_i \qquad 1 \le i \le n.$$

$$A \cdot \underset{\sim}{x}^T = \lambda \underset{\sim}{x}^T$$

Hence λ is an eigen-value of A.

$$Q(\underset{\sim}{x}) = \underset{\sim}{x} \cdot A \, \underset{\sim}{x}^T = \lambda \cdot \sum_{i=1}^{n} x_i^2 = \lambda .$$

Maximum value of $Q(\underset{\sim}{x})$ is the max. eigenvalue & the min. value of $(Q(\underset{\sim}{x})$ is the min. eigen-value.

Ex

Ex Find the max & min value of

$$Q = x^2 + y^2 + 2z^2 - 2xy + 4xz + 4yz,$$

subject to the constraint

$$x^2 + y^2 + z^2 = 1$$

$\underset{\sim}{x} = (x, y, z)$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$A \underset{\sim}{x}^T = \lambda \underset{\sim}{x}^T.$$

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 2 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} -1 & 1-\lambda \\ 2 & 2 \end{vmatrix} = 0$$

$$(1+\lambda)\left\{-2+\lambda^2-3\lambda\right\} + \left\{(\lambda-2)-4\right\} + 2\left\{-4+2\lambda\right\} = 0.$$

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0.$$

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$$(1-\lambda)\left\{-2+\lambda^2-3\lambda\right\} + \left\{(\lambda-2)-4\right\} + 2\left\{-4+2\lambda\right\} = 0.$$

4 is the max. eigen-value &

hence the max value of QF is 4

The max is attain at the pt. (x, y, z)

where

$$A\underset{\sim}{x}^T = 4\underset{\sim}{x}^T$$

$$(A - 4I)\underset{\sim}{x} = 0.$$

$$\begin{pmatrix} -3 & -1 & 2 \\ -1 & -3 & 2 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

$x = 1$
$y = 1$
$z = 2.$

The max is attained at the pt.

$$\frac{1}{\sqrt{6}}(1, 1, 2)$$