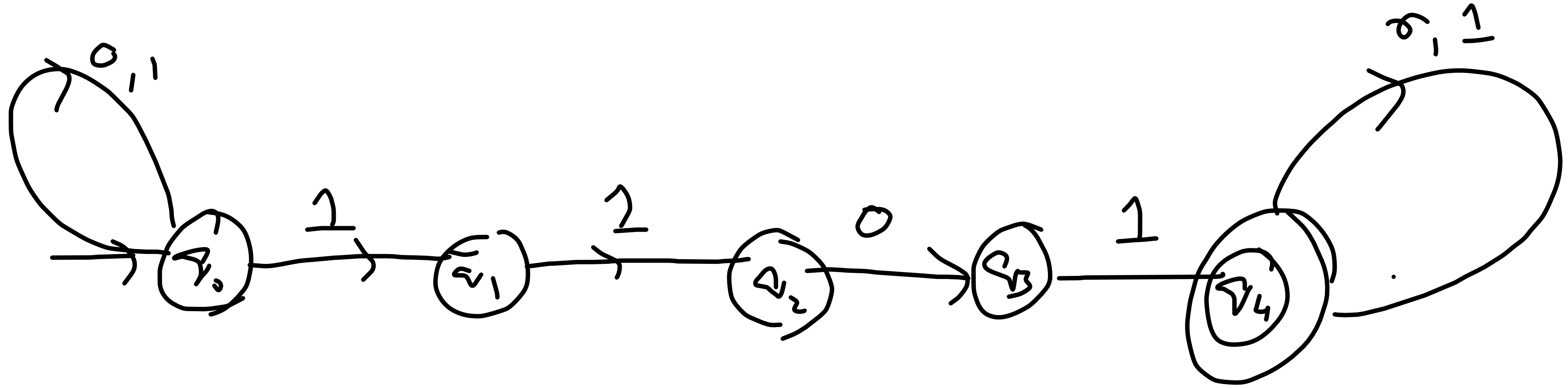
Ex The set of all strings over $\{0,1\}$ that contain 1101 as a substring is regular



Properties of regular langs.

<u>Thm</u> Regular langs. are closed under complementation, finite union & finite intersection.

Hence the class R of regular langs is a Boolean algebra.

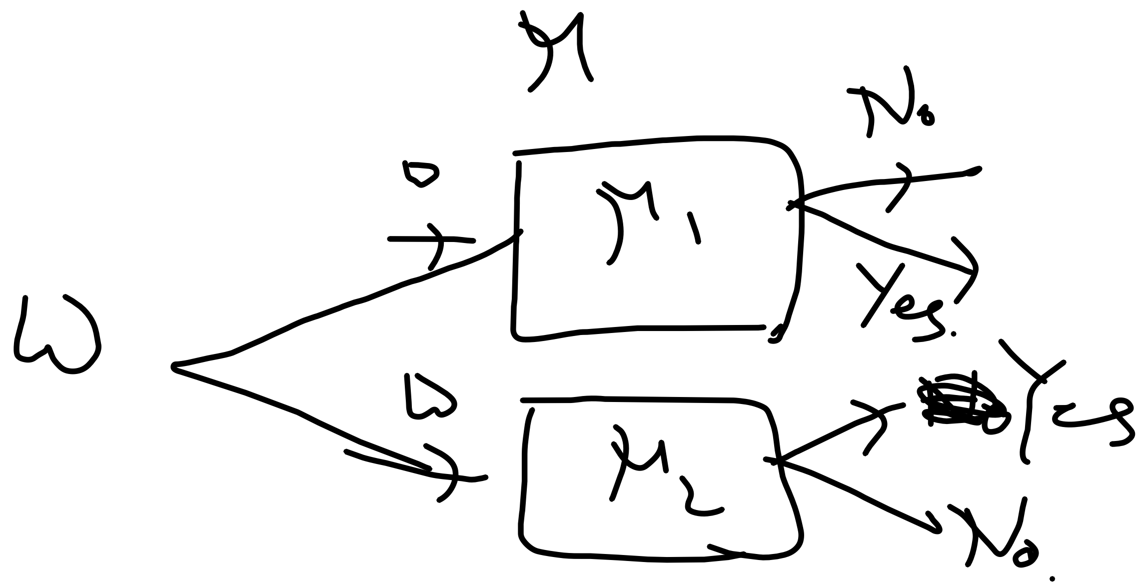
<u>Pf.</u> Clearly, the complement of a reg. lang is regular (Just interchange accepting & rejecting states.)

Closure under finite $\cap$.

Let $\mathcal{L}_1$ & $\mathcal{L}_2$ be two reg. langs. Fix two DFAs

$$M_1 = (\Sigma, Q_1, q_0^1, \delta_1, F_1)$$

$$M_1 = (\Sigma, Q_2, q_0^2, \delta_2, F_2)$$ accepting $\mathcal{L}_1$ & $\mathcal{L}_2$ respectively.



We shall construct a DFA $\hat{M}$ that will accepts $\mathcal{L}_1 \cap \mathcal{L}_2$

$$\hat{Q} = Q_1 \times Q_2$$

$$\hat{q}_0 = (q_0^1, q_0^2)$$

$$\hat{F} = F_1 \times F_2$$

Define

$$\hat{\delta} : \hat{Q} \times \Sigma \longrightarrow \hat{Q}.$$

$$\hat{\delta}((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)).$$

Claim. $\hat{\delta}^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w))$.

(Check this by induction on $|w|$).

$$w \in \mathcal{L}_1 \cap \mathcal{L}_2 \longleftrightarrow \delta_1^*(q_0^1, w) \in F_1 \ \& \ \delta_2^*(q_0^2, w) \in F_2$$

$$\longleftrightarrow \hat{\delta}^*((q_0^1, q_0^2), w) \in F_1 \times F_2 = \hat{F}.$$

$$\longleftrightarrow w \in \mathcal{L}(\hat{M}).$$

<u>Claim.</u> $\hat{\delta}^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w))$.

(Check this by induction on $|w|$).

$w \in \mathcal{L}_1 \cap \mathcal{L}_2 \longleftrightarrow \delta_1^*(q_0^1, w) \in F_1 \;\&\; \delta_2^*(q_0^2, w) \in F_2$

$\longleftrightarrow \hat{\delta}^*((q_0^1, q_0^2), w) \in F_1 \times F_2 = \hat{F}$.

$\longleftrightarrow w \in \mathcal{L}(\hat{M})$.

Thm Regular langs are closed under concatenation & Kleene closure.

pf (a). Fix 2 regular langs $\mathcal{L}_1$ & $\mathcal{L}_2$. Fix 2 DFAs M_1 & M_2 accepting $\mathcal{L}_1$ & $\mathcal{L}_2$ respectively.

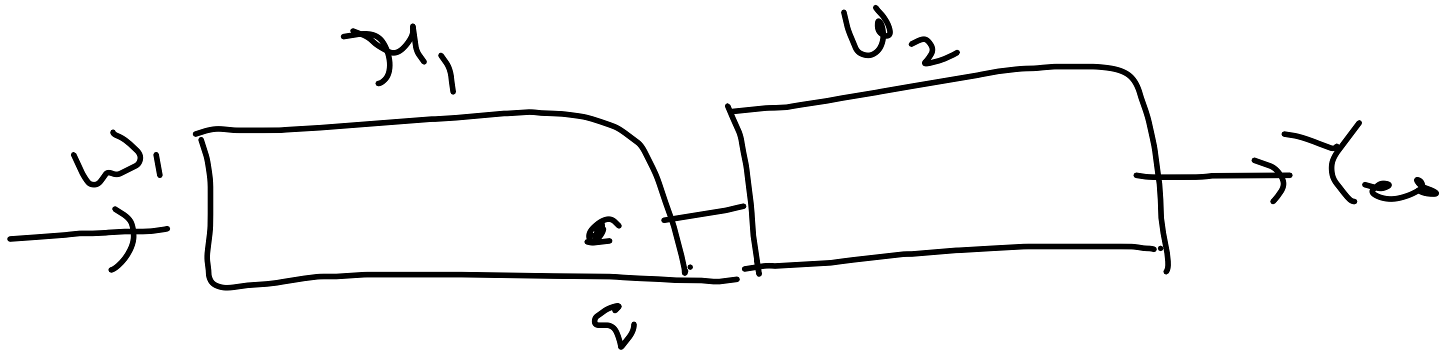
Let $M_1 = (\Sigma, Q_1, q_0^1, \delta_1, F_1)$

Thm Regular langs are closed under concatenation & Kleene closure.

Pf ⓐ. Fix 2 regular langs L_1 & L_2. Fix 2 DFAs M_1 & M_2 accepting L_1 & L_2 respectively.

Let $M_1 = (\Sigma, Q_1, q_0^1, \delta_1, F_1)$

& $M_2 = (\Sigma, Q_2, q_0^2, \delta_2, F_2)$



We shall construct an NFA $\hat{M} = (\Sigma, \hat{Q}, \hat{q}_0, \hat{\delta}, \hat{F})$

that accepts $L_1 * L_2$

$$\hat{Q} = Q_1 \cup Q_2$$

$$\hat{q}_0 = q_0^1, \quad \hat{F} = F_2$$

Define $\hat{\delta}$ as follow.

$$\hat{\delta}(q, a) = \begin{cases} \{\delta_1(q, a)\} & \text{if } q \in Q - F \\ \{\delta_1(q, a), \delta_2(q_0^2, a)\} & \text{if } q \in F \\ \{\delta_2(q, a)\} & \text{if } q \in Q_2 \end{cases}$$

Check that $\mathcal{L}(\hat{M}) = \mathcal{L}_1 * \mathcal{L}_2$

Closure under *.

Fix a DFA $M = (\Sigma, Q, q_0, \delta, F)$ that accepts $\mathcal{L}$.

We shall construct an NFA $\hat{M} = (\Sigma, \hat{Q}, \hat{q}_0, \hat{\delta}, \hat{F})$ that accepts $\mathcal{L}^*$

$$\hat{Q} = Q \cup \{q^*\}$$

$$\hat{q}_0 = q^*.$$

$$\hat{F} = F \cup \{q^*\}$$

$$\hat{\delta}(q, a) = \begin{cases} \{\delta(q, a)\} & \text{if } \delta(q, a) \notin F \\ \{\delta(q, a), q^*\} & \text{if } \delta(q, a) \in F \\ \{\delta(q_0, a)\} & \text{if } q = q^* \end{cases}$$

Check that $\mathcal{L}(\hat{\eta}) = \mathcal{L}^*$.

Ex Let $\mathcal{L}_1$ & $\mathcal{L}_2$ be regular langs. Show that

$$\mathcal{L}_1 / \mathcal{L}_2 = \{ u \in \Sigma^* : \text{for some } v \in \mathcal{L}_2,\ uv \in \mathcal{L}_1 \}.$$

is also regular.

What is $\mathcal{L}_1 / \mathcal{L}_2$ if $\mathcal{L}_2 = \Sigma^*$? $\quad \mathrm{Init}(\mathcal{L}_1)$

Regular expressions over Σ.

(a) For each $a \in \Sigma$, $\underset{\sim}{a}$ is a regular expⁿ & represents the set $\{a\}$

Also $\underset{\sim}{\lambda}$ is a regular expⁿ that represents $\{\lambda\}$

(b) If α and β are regular expⁿ representing L_1 & L_2 (resp)

Then $(\alpha+\beta)$ is also a regular expⁿ repⁿ $L_1 \cup L_2$

(c) If α & β . . . . . expⁿ . . . . . L_1 & L_2 then

so is $(\alpha . \beta)$ & represents $L_1 . L_2$

(d) If α is a regular expression rep^n $\mathcal{L}$
Then so is (α^*) representing $\mathcal{L}^*$.

<u>Remark.</u> $\mathcal{L}(\alpha)$ denote that lang. represented by α.

<u>Ex</u> $(\underline{0} + \underline{1})^* =$ all binary strings.

$(\underline{0} + \underline{1})^* \underline{0}$

$(\underline{1}\underline{1}\underline{0}\underline{1})^* . (\underline{0} + \underline{1})^*$.

Properties of regular $\exp^n$.

(a) $\alpha^{**} = \alpha^*$ (b) $\alpha + \beta = \beta + \alpha$.

(c) $\alpha . (\beta . \gamma) = (\alpha . \beta) . \gamma$.

(d) $\alpha . (\beta + \gamma) = \alpha . \beta + \alpha . \gamma$.

$(\alpha + \beta) . \gamma = \alpha . \gamma + \beta . \gamma$.

(e) $\alpha . \alpha^* + \lambda = \alpha^*$

(f) $(\alpha + \beta)^* = (\alpha^* \beta^*)^* = (\alpha^* + \beta^*)^*$

Properties of regular $\exp^n$.

(a) $\alpha^{**} = \alpha^*$ (b) $\alpha + \beta = \beta + \alpha$.

(c) $\alpha . (\beta . \gamma) = (\alpha . \beta) . \gamma$.

(d) $\alpha . (\beta + \gamma) = \alpha . \beta + \alpha . \gamma$.

$(\alpha + \beta) . \gamma = \alpha . \gamma + \beta . \gamma$.

(e) $\alpha . \alpha^* + \lambda = \alpha^*$

(f) $(\alpha + \beta)^* = (\alpha^* \beta^*)^* = (\alpha^* + \beta^*)^*$

pf of (f). $\left((\alpha+\beta)^*\right) \subseteq \left((\alpha^*\beta^*)^*\right)$

$$\alpha \subseteq \alpha^*\beta^*.$$

||ly $\beta \subseteq \alpha^*\beta^*$

$$\alpha+\beta \subseteq \alpha^*\beta^* + \alpha^*\beta^* = \alpha^*\beta^*.$$

$$\Longrightarrow (\alpha+\beta)^* \subseteq (\alpha^*\beta^*)^*.$$

$$\mathcal{L}((\alpha^*\beta^*)^*) \subseteq \mathcal{L}((\alpha^*+\beta^*)^*)$$

We have

$$\alpha^* \subseteq \alpha^*+\beta^*$$
$$\beta^* \subseteq \alpha^*+\beta^*.$$

$$\alpha^*.\beta^* \subseteq (\alpha^*+\beta^*).(\alpha^*+\beta^*) \subseteq (\alpha^*+\beta^*)^*.(\alpha^*+\beta^*)^*.$$
$$= (\alpha^*+\beta^*)^*.$$

$$\Rightarrow (\alpha^*.\beta^*)^* \subseteq (\alpha^*+\beta^*)^{**}$$
$$= (\alpha^*+\beta^*)^*.$$

$$\mathcal{L}((\alpha^* + \beta^*)^*) \subseteq \mathcal{L}((\alpha + \beta)^*).$$

$$\alpha^* \subseteq (\alpha + \beta)^*.$$

$$\beta^* \subseteq (\alpha + \beta)^*.$$

$$\Rightarrow \alpha^* + \beta^* \subseteq (\alpha+\beta)^* + (\alpha+\beta)^* = (\alpha+\beta)^*.$$

$$(\alpha^* + \beta^*)^* \subseteq (\alpha+\beta)^{**} = (\alpha+\beta)^*$$

Kleene's Thm

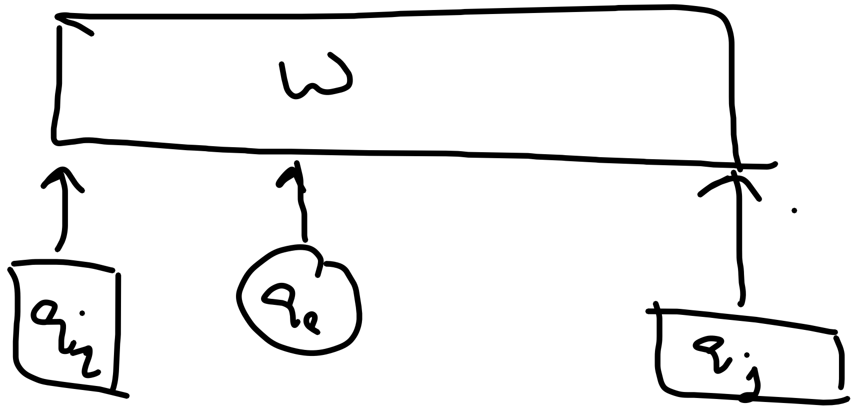
A lang. $\mathcal{L}$ is regular iff there is a regular exp$^{\underline{n}}$ α s.t. $\mathcal{L}(\alpha) = \mathcal{L}$.

pf $\leftarrow$ If α is a regular exp$^{\underline{n}}$ then $\mathcal{L}(\alpha)$ is regular follows from def$^{\underline{n}}$ & the closure properties

$\rightarrow$ Let $\mathcal{L}$ be a regular lang.

Fix a DFA $M = (\Sigma, Q, q_1, \delta, F)$ that accepts L.

Let $Q = \{q_1, q_2, \ldots, q_n\}$



Define, for $1 \le i, j \le n$, $0 \le k \le n$

$R^k_{i,j} = \{ w \in \Sigma^* :$ M in state q_i on reading w enters state q_j & M does not enter any state q_ℓ with $\ell > k$. $\}$

What is $R^0_{i,j}$?

$$R^0_{i,j} = \{ a \in \Sigma : \delta(q_i, a) = q_j \} = \{a, b, c\}$$

$$\underline{a} + \underline{b} + \underline{c}$$

& hence finite. Hence it can be represented by a regular exp$^{\underline{n}}$

$$\mathcal{L} = \bigcup_{q_j \in F} R^{n}_{1,j}.$$

We are done if we can show that each $R^{k}_{i,j}$ can be represented by a regular expr^{n}