Institute for Advancing Intelligence (IAI), TCG-CREST
Admission Test
Ph.D Program Session: 2022–2023

Date: 28. 05. 2022 Time: $3\frac{1}{2}$ Hours, 2PM to 5:30PM

GROUP A: General Mathematics

1. An n-variable Boolean function is a function of the form $f : \{0,1\}^n \to \{0,1\}$. The weight of an element $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$, denoted by $wt(x_1, x_2, \ldots, x_n)$, is defined as the number of 1's in the n-bit tuple. A Boolean function is called symmetric if $f(x_1, x_2, \ldots, x_n) = f(y_1, y_2, \ldots, y_n)$, whenever $wt(x_1, x_2, \ldots, x_n) = wt(y_1, y_2, \ldots, y_n)$, for $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \{0,1\}^n$.

   (a) Consider the following functions and explain whether they are symmetric or not.
      (i) $f_1(x_1, x_2) = x_1 \cdot x_2$ ($\cdot$ means logical AND)
      (ii) $f_2(x_1, x_2, x_3) = x_1 \oplus x_2$ ($\oplus$ means logical XOR)
      (iii) $f_3(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$

   (b) Prove that all the single variable Boolean functions are symmetric.

   (c) Deduce the number of distinct n-variable symmetric Boolean functions.

   (d) Let $f_4(x_1, x_2, \ldots, x_n) = f_5(x_1, x_2, \ldots, x_n) \oplus x_1 \cdot x_2 \cdot \ldots \cdot x_n$. Prove that f_4 is symmetric if and only if f_5 is symmetric.

   [(2+2+2) + 4 + 6 + 4 = 20]

2. (a) Let I be an $n \times n$ identity matrix and J be an $n \times n$ matrix with all entries 1. (i) Compute J^2. (ii) Find conditions on the real values of a and b such that $aI + bJ$ is invertible. Compute the inverse when $aI + bJ$ is invertible.

   (b) Let $(G, +)$ be a group of size N, not necessarily commutative. Find the number of solutions in $(x_1, x_2, x_3, x_4) \in G \times G \times G \times G$ satisfying the following equations: $x_1 + x_2 = x_3,\ x_2 + x_3 = x_4,\ x_3 + x_4 = x_1$.

   [(2+8) + 10 = 20]

3. (a) A message is a sequence of bits, that is, 0s and 1s. Transmitting the bit 0 takes one unit of time while sending the bit 1 takes two units of time. Now transmitting different messages might incur the same time; for example, the messages 000, 01, or 10 will take exactly three units of time.
      (i) Enumerate all the messages that can be sent in exactly *five* units of time.
      (ii) Obtain the number of messages that can be sent in exactly T units of time.

   (b) A directed graph $G = (V = \{v_1, \ldots, v_n\}, E)$ is called "*any-directional*" if

   $$\text{for all } i < j,\ (v_j, v_i) \in E \text{ if } (v_i, v_j) \notin E.$$

      (i) Draw all possible *any-directional* simple graphs with two vertices.
      (ii) Count the number of n-vertices *any-directional* simple graphs.

(iii) Count the number of n-vertices *any-directional* graphs if we allow self loops.

[Note: A graph is called simple if it doesn't contain any self-loop or multiple edges.]

[(3+7) + (2+5+3) = 20]

4. (a) Let n be a positive integer greater than or equals to 6. Consider an n-sided convex polygon $A_1 A_2 \ldots A_n$.
   (i) Prove that there can be at most five interior angles which are less than 120.
   (ii) Show that there exists $1 \leq i < j \leq n$ such that

$$|\cos A_i - \cos A_j| \leq \frac{1}{2(n-6)}.$$

   [Note: $\cos 120 = -1/2$, $\cos 180 = -1$.]

   (b) You are given three biased coins. For the first two coins, the probability of getting a head is 2/3, whereas the probability of getting a head for the third coin is 7/8. Design an unbiased coin that uses exactly one toss of each of these three coins. [Note: Essentially you have to define an event whose probability is 1/2.]

[(5+5) + 10 = 20]

GROUP B: Technical Topics related to Computer Science

5. Consider an inter-college chess competition where each player plays exactly once with everyone else. Assume each match has a winner, tie-breakers are used for matches that results in a tie. A player P can become a champion if for all the other players Q, (i) either P beats Q, or (ii) there exists another player R such that P beats R and R beats Q.

   (a) Show that the competition has at least one champion.
   (b) Justify whether it is possible for a player to become one of the champions of the competition if he/she wins only a single match.
   (c) Assuming the number of participants in the competition to be odd, is it possible that all the participants of the competition become a champion? Justify. [Hint: Check with a small example and then try to use induction.]

[8 + 4 + 8 = 20]

6. A k-tree is defined as a tree with each of it's non-leaf nodes containing exactly k children.

   (a) Obtain the minimum and maximum height of a 2-tree with n nodes. Take an example with a suitable n, such that $n \geq 6$ and demonstrate both the cases for n nodes.
   (b) Generalize the previous result for a k-tree, that is, obtain the minimum and maximum height of a k-tree with n nodes.

[10 + 10 = 20]

7. Suppose you are given n points on the curve $y = x^2 + 1$, sorted according to their X co-ordinate values.

   (a) Report the maximum Y co-ordinate value in *constant time*.

   (b) Write a *logarithm time* algorithm to report the point with the minimum Y co-ordinate value.

   (c) Design a *linear-time* algorithm that sorts the n points according to their Y co-ordinate values. Demonstrate your algorithm with an example having more than seven input points.

[2 + 8 + 10 = 20]

8. An institute is going to recruit students based on their marks obtained in the admission test. Each candidate is designated by a pair (r_i, m_i), where r_i is the unique roll number and m_i is the marks of the i^{th} candidate.

   (a) Suppose the institute decides to recruit the top k candidates. Given an unsorted array X of (roll number, marks) pairs and a positive integer $k \leq |X|$, describe an $O(|X| + k \log |X|)$-time algorithm to return the roll numbers of the k best candidates according to the marks obtained.

   (b) Suppose the institute changes the policy and decides to recruit all the candidates having marks strictly greater than a threshold t. Suppose all the data are stored in a max-heap. Describe an $O(n_t)$-time algorithm to return the roll numbers of all applicants with marks larger than t, where n_t is the number of candidates returned.

[10 + 10 = 20]

9. (a) In Figure 1, R1 = 2 kΩ, R2 = 3 kΩ, R3 = 6 kΩ and R4 = 9 kΩ. The impedance Z can be either a resistance R, or a capacitor C or an inductor L, or any combination of those (including the open and short circuits). Calculate the value of steady state equivalent impedance Z_{eq} for each of the cases, given any DC voltage source.

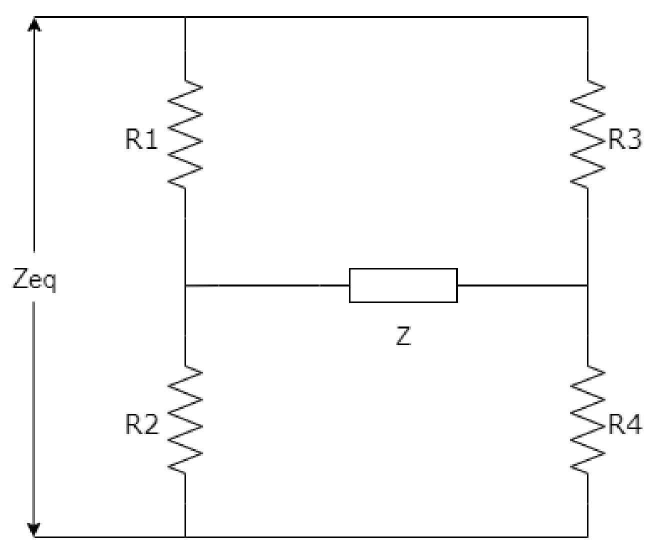

Figure 1:

   (b) Consider the equation $f(n-1)(f(n) - a_n) = 1$, for $n > 1$, and $f(1) = a_1$. Now fix an n. Describe a circuit using resistances, voltage source and an Ammeter as well as a Voltmeter, so that $f(n)$

can be computed for any set of finite positive values $a_1, a_2, \ldots, a_n$. You may first explain with a specific example for $n = 4$.

[10 + 10 = 20]

10. (a) A bin consists of four 0.47 μF, three 4.7 μF and three 47 μF capacitors (total ten). Two capacitors C_1, C_2 are picked up together uniformly at random, and connected in parallel, i.e., $C_1||C_2$. What are the possible distinct values of the equivalent capacitors? Plot $\Pr(C_1||C_2 = x)$ vs x, for $0 \leq x \leq 100$.

(b) We require to construct a Boolean circuit for a function $f(x_1, x_2, \ldots, x_n) = x_1 \oplus x_2 \oplus \ldots \oplus x_n$ with the condition that only two input XOR ($\oplus$) gates are available, where the delay of each gate is t ns (nanosecond). How can you implement the function with minimum number of two input gates, achieving minimum total delay (you need to logically define the total delay of the circuit)? Provide an algorithmic outline for the circuit synthesis and demonstrate for $n = 10, t = 2$.

[10 + 10 = 20]

GROUP C: Technical Topics related to Mathematics

11. (a) Let (a_n) and (b_n) be the sequence of real numbers such that $|a_n + b_n| > \sqrt{2} - sin(\frac{1}{n})$ and ${a_n}^2 + {b_n}^2 = 1$. Show that $|a_n + b_n|$ converges to $\sqrt{2}$.

(b) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that φ', φ'' exist and are continuous on $[1, 2]$ with $\varphi(x)$, $\varphi'(x)$ vanishing at $x = 1, 2$. Prove that there exists a constant $c > 0$ such that for any $\lambda > 1$,

$$\left| \int_1^2 e^{i\lambda x} \varphi(x) dx \right| \leq \frac{c}{\lambda^2}.$$

(c) Find the solution $y(t)$ of the initial-value problem

$$e^y \frac{dy}{dt} - (t + t^3) = 0, \quad y(1) = 1.$$

[8 + 8 + 4 = 20]

12. (a) Let

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 & 2x \\ 3 & 3 & 3 & 3x & 3 \\ 4 & 4 & 4x & 4 & 4 \\ 5 & 5x & 5 & 5 & 5 \\ 6x & 6 & 6 & 6 & 6 \end{pmatrix}$$

Find the determinant of A in terms of x using row and/or column operations. Determine all the values of x for which A is not invertible.

(b) Let A be a 50×50 matrix with real entries such that $\text{rank}(A) = 50$. Find $\text{rank}(A^{50})$ and justify your answer.

(c) Prove that the matrix $A = \begin{pmatrix} 0 & 3 & 1 & 0 \\ 3 & 0 & 3 & 0 \\ 1 & 3 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}$ has two positive and two negative eigenvalues (counting multiplicities).

[10 + 5 + 5 = 20]

13. A real number $\alpha \in \mathbb{R}$ is constructible if and only if there is a tower of subfields of $\mathbb{R}$

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \subseteq \mathbb{R}$$

where each $F_i = F_{i-1}(\sqrt{a_i})$ for some $a_i \in F_{i-1}$ such that $\alpha \in F_k$.

(a) Show that the polynomial $y^3 - 3y - 1$ is an irreducible polynomial over $\mathbb{Q}$.

(b) Let $a = \cos 20°$. Show that $a \notin \mathbb{Q}$.
[Hint: Use the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ for any angle θ.]

(c) Use part (a) to show that a is not constructible.

[6 + 7 + 7 = 20])

14. (a) Let $\{g_n\}$ be a sequence of twice differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $g_n(0) = g_n'(0) = 0$ for all n. Assume that $|g_n''(x)| \leq 1$ for all n and all $x \in [0,1]$. Prove that

(i) $|g_n(x)| \leq \frac{1}{2}, \forall x \in [0,1]$.

(ii) $|g_n(x) - g_n(y)| \leq |x - y|, \forall x, y \in [0,1]$

(b) Let $f = u + iv$ be analytic in connected open set D, where u and v are real valued functions. Suppose there are real constants $a,\ b,$ and c such that $a^2 + b^2 \neq 0$ and $au + bv = c$ in D. Prove that f is a constant function.

[10 + 10 = 20]

15. (a) Let

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\},$$

and

$$D^{n+1} = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 \leq 1\}$$

be respectively the n-sphere and the $(n+1)$-disc in $\mathbb{R}^{n+1}$, $n \geq 1$, considered as subspaces of $\mathbb{R}^{n+1}$. Let $f : S^2 \to D^2$ be any continuous function. Prove that f is homotopic to a constant map.

(b) Let (X, d) be a metric space. Let A and B be compact subsets of X such that $A \cap B = \emptyset$. Prove that there exists disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

[10 + 10 = 20]

16. Prove that any finite group of order n is isomorphic to a subgroup of $O(n)$, the group (with respect to matrix multiplication) of $n \times n$ orthogonal real matrices. [Hint: Consider the group as a subgroup of the Symmetric group S_n.] [20]