Order Statistics

A → n element Array

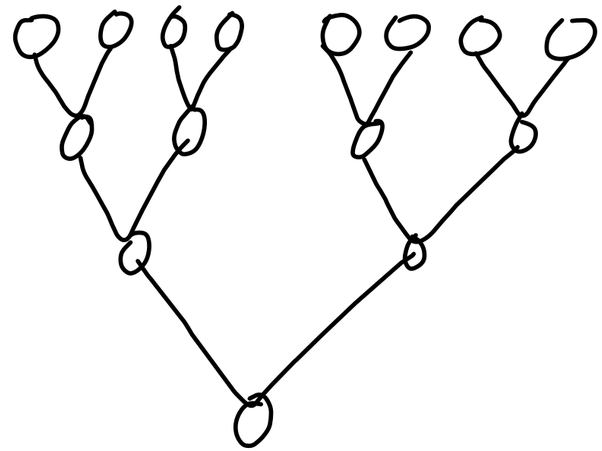
① How many comparisons are required to find the minimum element?

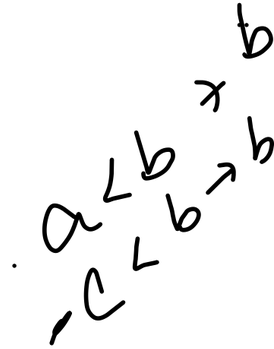
$(n-1)$

$A[a] \overset{?}{<} A[b]$

↓

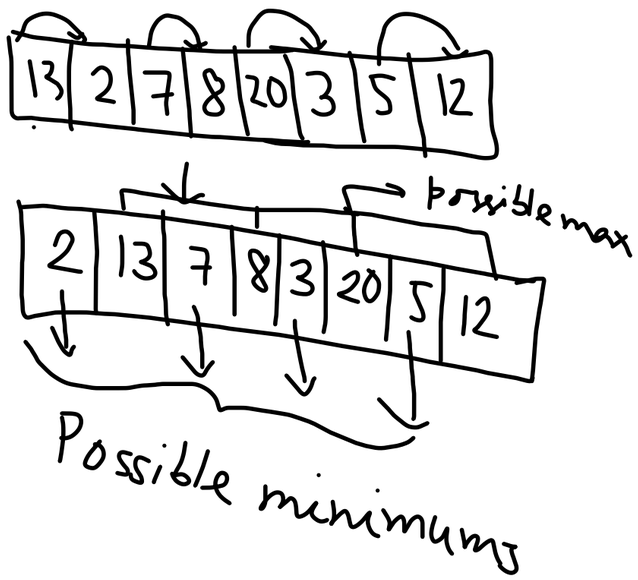
$a < b$ → Remove 1 element.





② How many comparisons are required to find both minimum & maximum?

Trivial (used in ①) $\Rightarrow (n-1)+(n-1) = 2n-2$.



$\frac{n}{2}$

$\left(\frac{n}{2}-1\right)$

$\left(\frac{n}{2}-1\right)$

$\left(\frac{3n}{2}-2\right)$

If $A[2i] > A[2i+1]$

Swap $(A[2i], A[2i+1])$.

$a_1 < a_2$

↓

- a_2 can't be min^m
- a_1 can't be max^m

- Prove this is the min^m no. of comparisons required

③ How many comparisons are required to report an element which is not the second minimum?

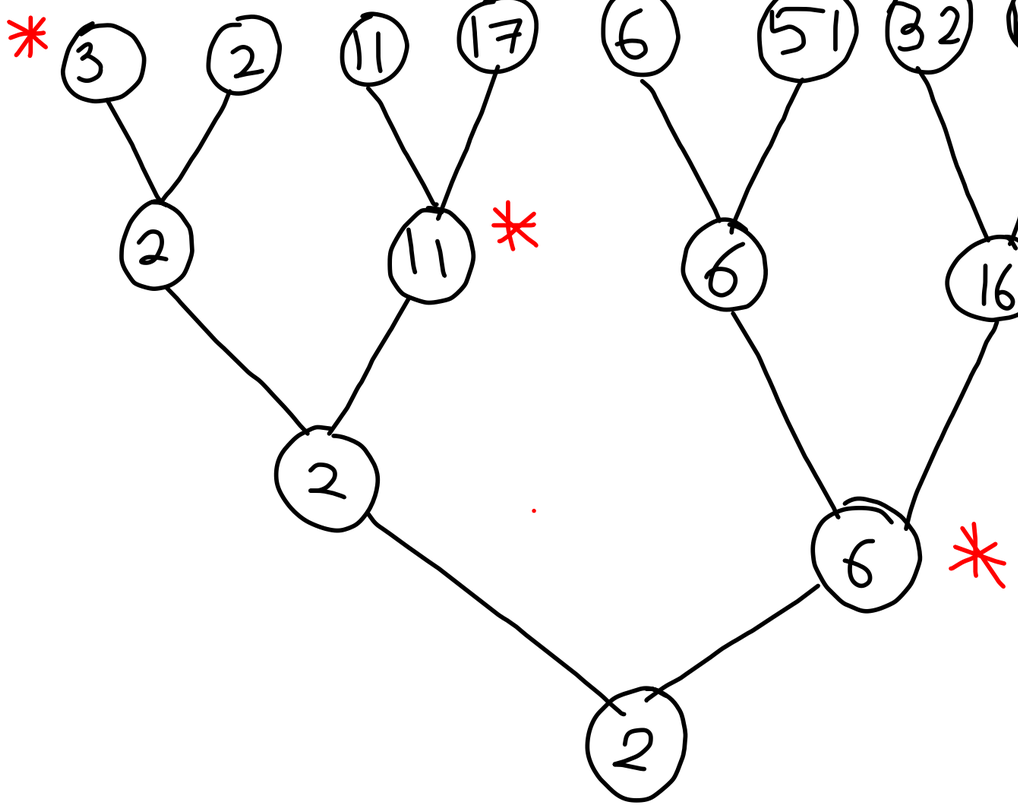
1. $A[1] < A[2]$
2. $A[2] > A[3]$
3. $A[1] < A[3]$

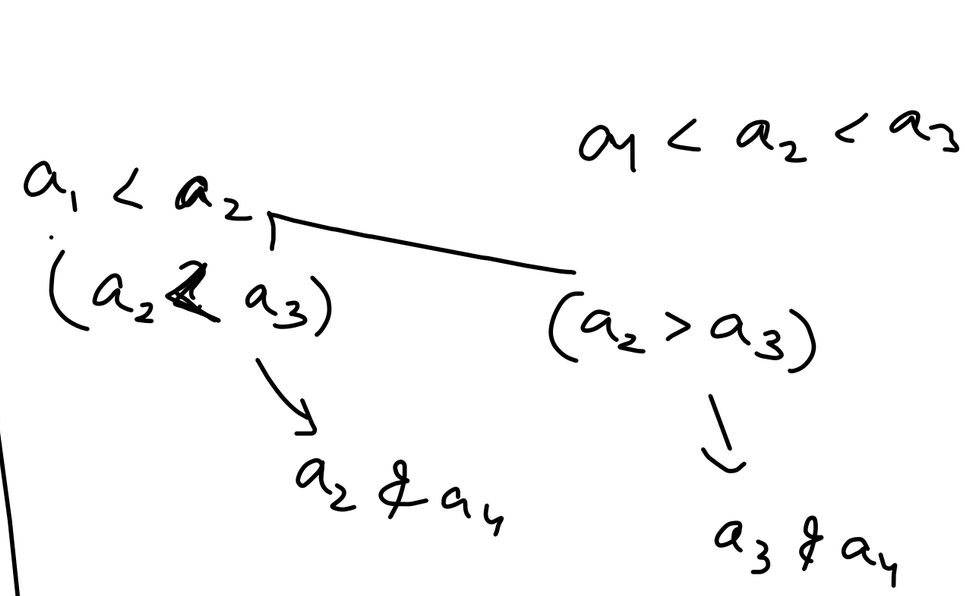
$A[2]$ is not 2nd min.

$A[1] < A[2]$ — Take the 1st max

$A[2] < A[3]$ — Then compare that with the 3rd element

(4) Finding both Minimum & Second Minimum?





$$(n-1) + (\log n - 1)$$
$$= n + \log n - 2$$

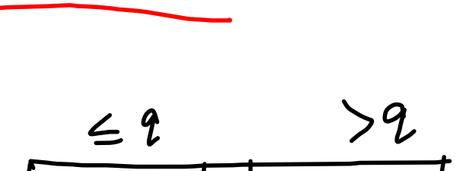
Finding i^{th} minimum element in $O(n)$ worst-case

Complexity $T(n)$

- Divide the elements in $\lceil \frac{n}{5} \rceil$ many subarrays each containing 5 elements
- Sort each subarray $O(n)$
- Find median of medians of the subarrays $T(\frac{n}{5})$
- Partition on that element as pivot in Selection algo recursively. $T(\frac{7n}{10})$

$$T(n) = T(\tfrac{9n}{10}) + c \cdot n$$
$$\downarrow$$
$$T(n) = O(n)$$

$$T(n) = T(\tfrac{n}{5}) + T(\tfrac{7n}{10}) + O(n)$$
$$\Rightarrow T(n) = O(n), \; n > 140$$



$$T(n) = \max\{T(k), T(n-k)\} + O(n)$$

$k \approx \frac{n}{2}$ would be perfect

If $k = 1/n$ → $T(n) = O(n^2)$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + c_1 \cdot n$$

$$= cn.$$

$$T(n) = c\frac{n}{5} + c \cdot \frac{7n}{10} + c_1 \cdot n$$

$$= \left(cn\frac{9}{10} + c_1 n\right)$$

$$cn = cn\frac{9}{10} + c_1 n$$

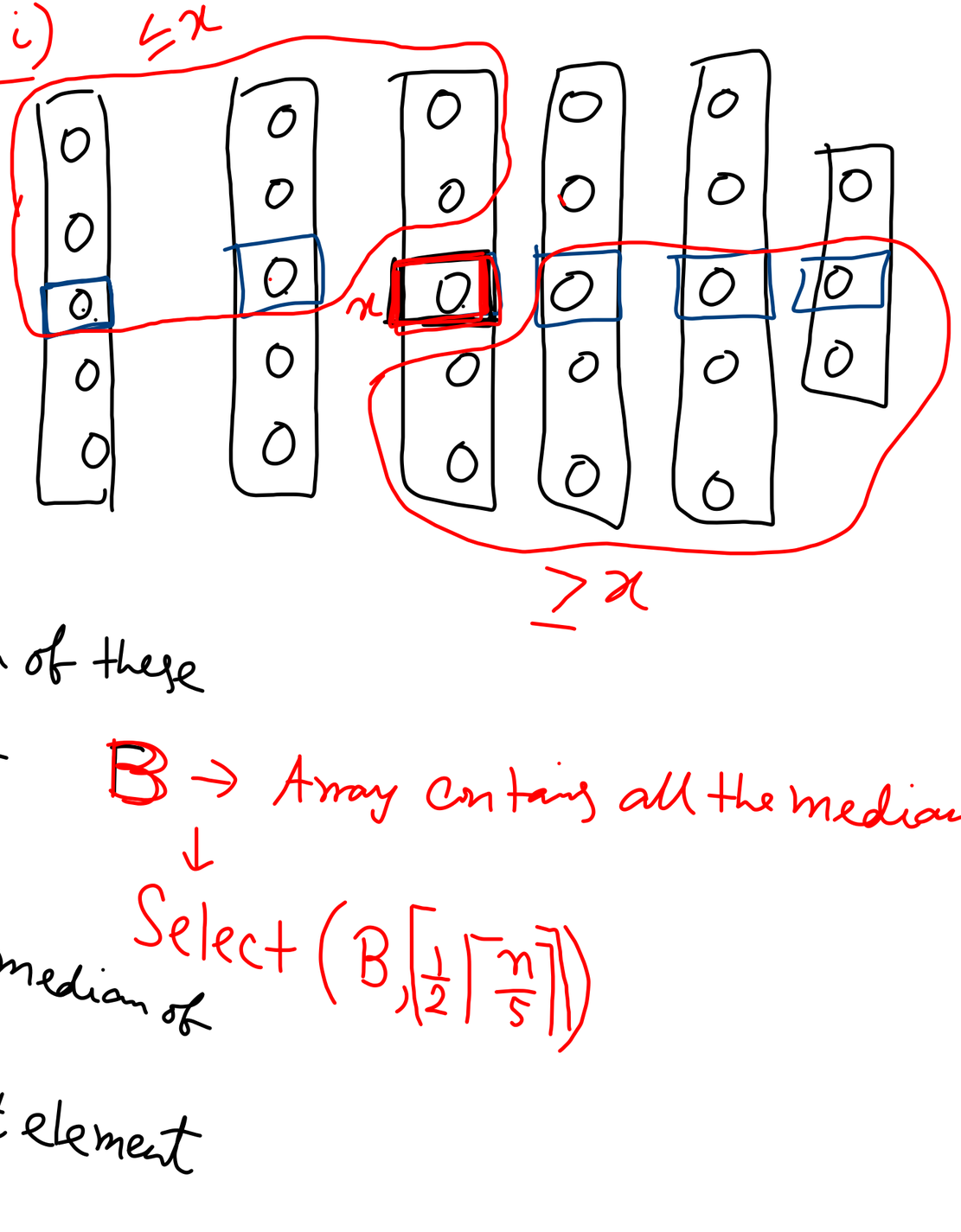
$$\frac{c}{10} = c_1 \qquad c = 10c_1$$

Select(A, i)

- Sort all individual arrays

↓

$O(n)$



- Find median of these medians -

B → Array contains all the medians

↓

Select $\left(B, \left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil\right)$

- Let x be median of medians
- x ⇒ pivot element

Divide n into $\lceil \frac{n}{5} \rceil$ sub-arrays each containing 5 elements & one containing ≤ 5 elements

#elements ≥ x atleast

⇓

$\left(3 \cdot \left\lceil \frac{n}{10} \right\rceil - 2\right)$

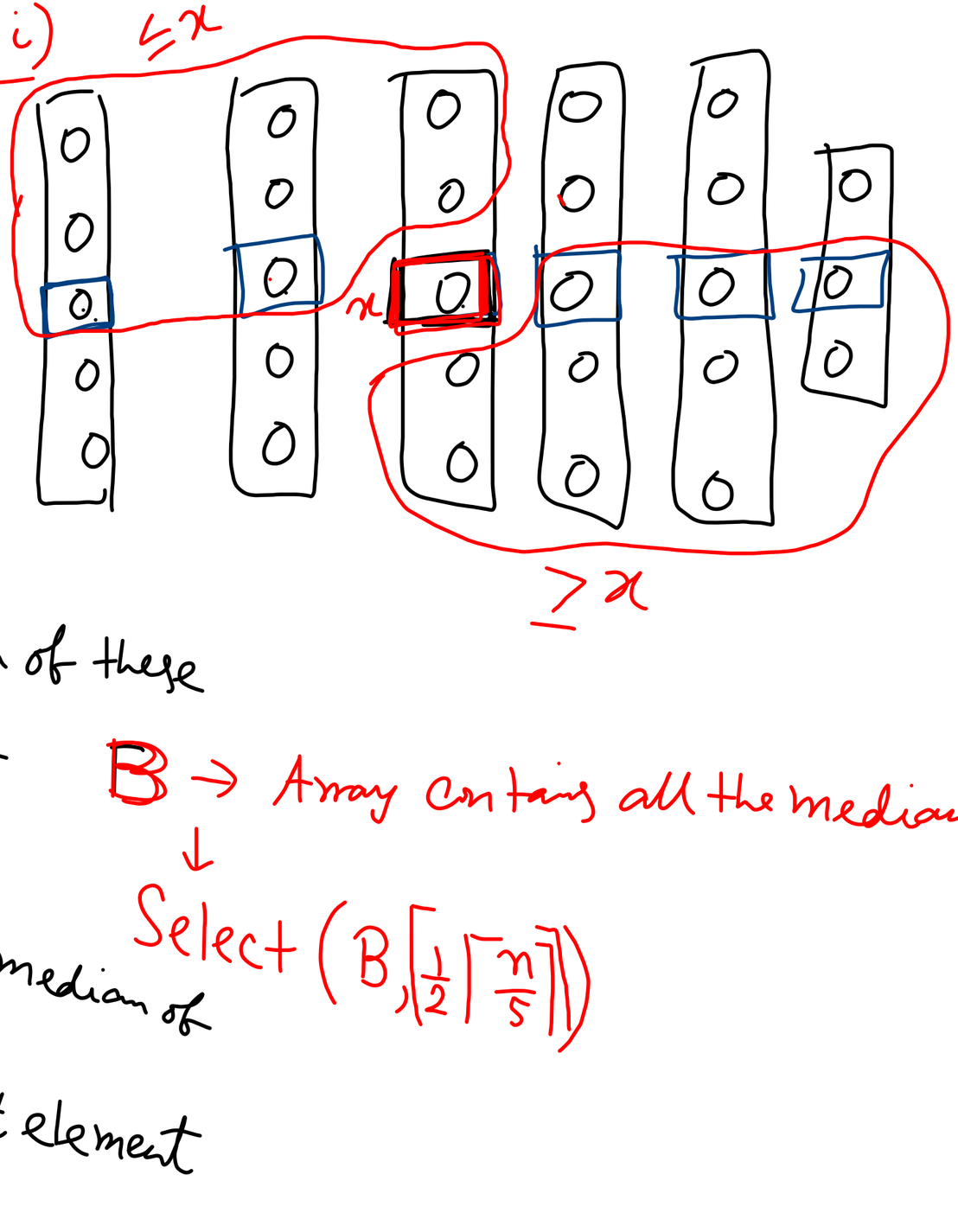
at most $\approx 7 \left\lceil \frac{n}{10} \right\rceil$

Select(A, i)

- Sort all individual arrays

↓

$O(n)$



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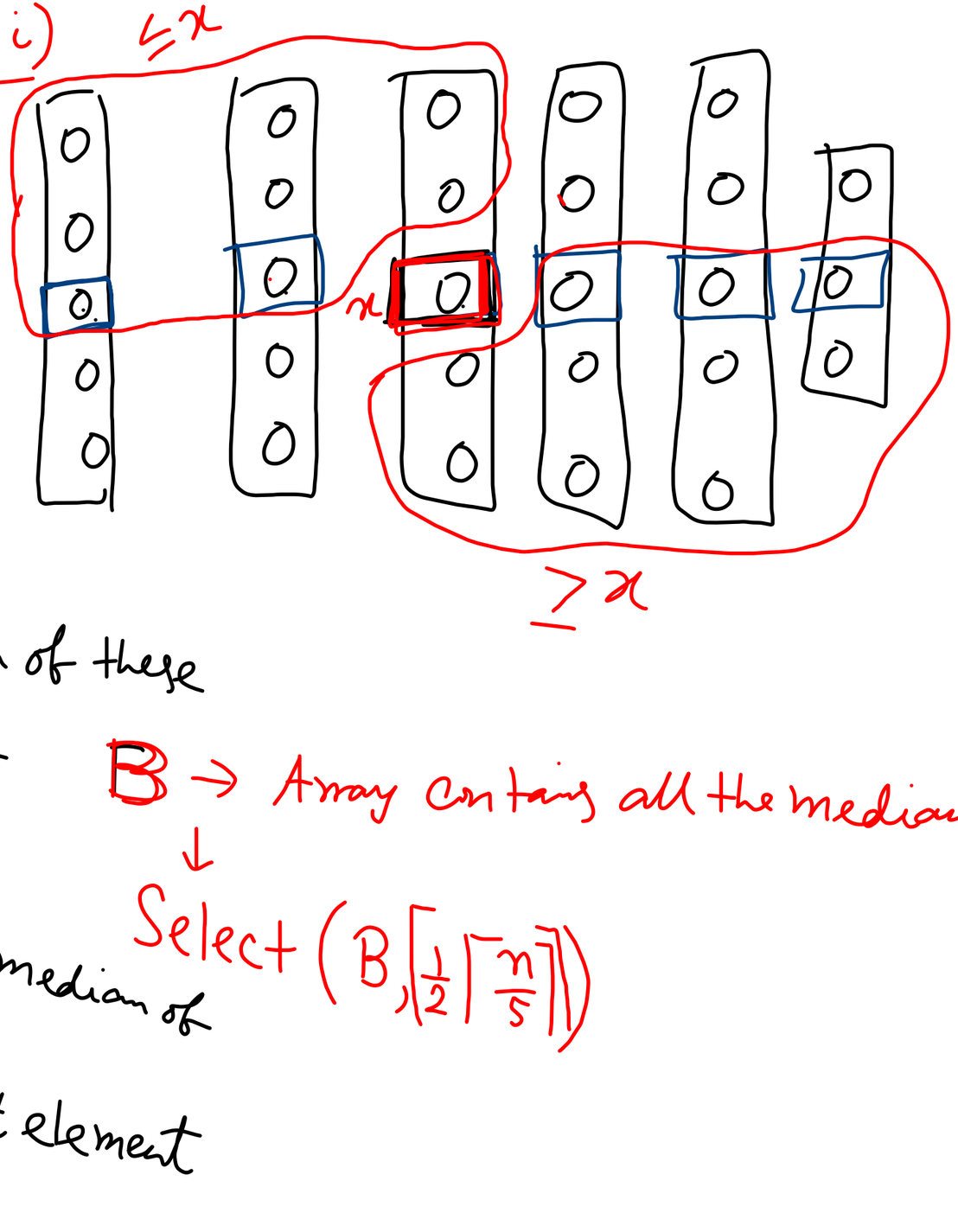
$O(n)$



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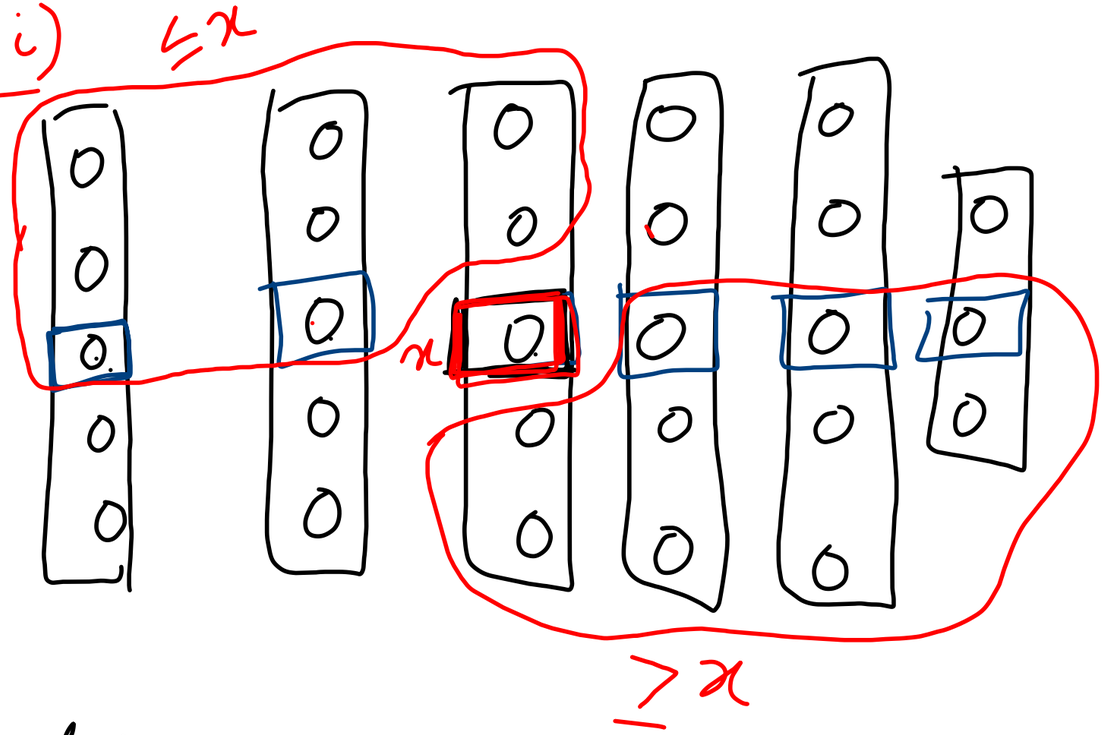

- Find median of these medians -

$B \rightarrow$ Array contains all the medians

↓

$\text{Select}\left(B, \left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil\right)$

- Let x be median of medians
- $x \Rightarrow$ pivot element

Divide n into $\lceil \frac{n}{5} \rceil$ sub-arrays each containing 5 elements & one containing ≤ 5 elements

#elements $\geq x$ atleast

$\Downarrow$

$\left(3 \cdot \left\lceil \frac{n}{10} \right\rceil - 2\right)$

at most $\approx 7 \left\lceil \frac{n}{10} \right\rceil$