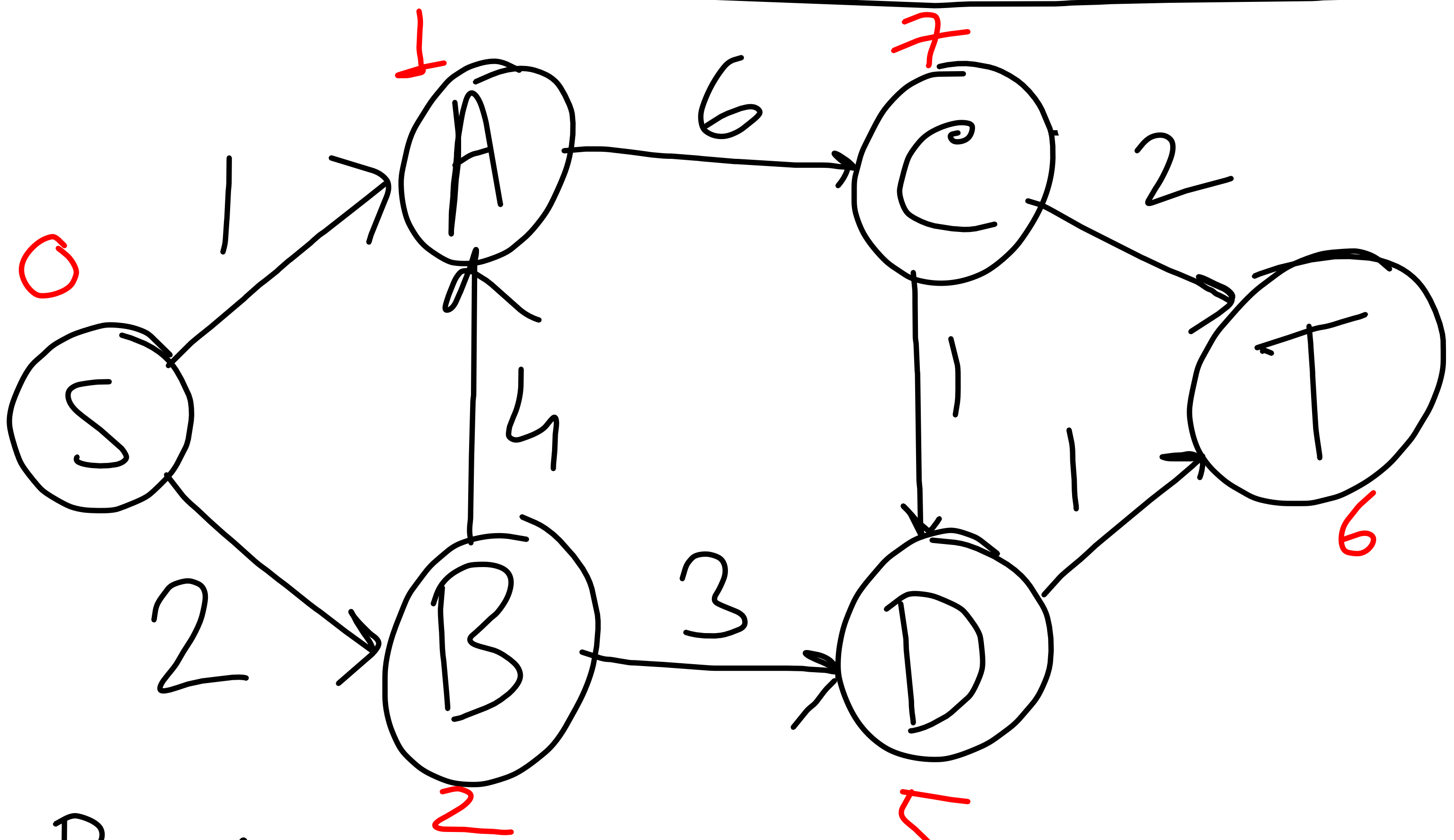
Directed Acyclic Graph (DAG)

$\text{dist}(S) = 0$



S, B, A, C, D, T

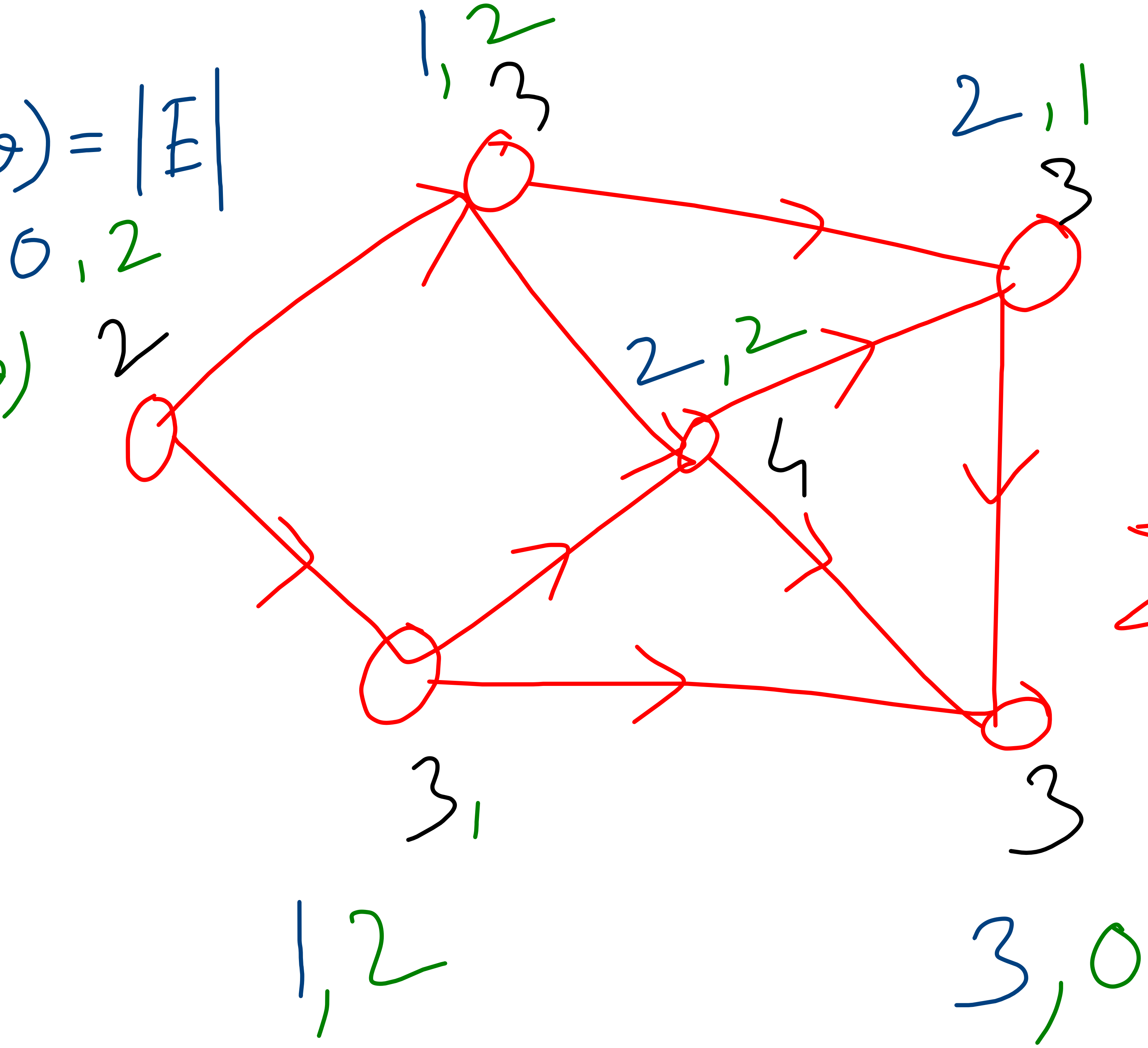
$$\text{dist}(D) = \min\{\text{dist}(C)+1, \text{dist}(B)+3\}$$

$$= \min\{7+1, 2+3\}$$

$$= 5$$

$$\sum \text{indegree}(v) = |E|$$

$$\sum \text{outdegree}(v) = |E|$$



$$|V| = 6$$

$$|E| = 9$$

$$\sum \deg(v) = 18 = 2 \times |E|$$

$\text{dist}(s) = 0$

for $v \in V$ in Linear order

$$\text{dist}(v) = \min_{(u,v)\in E} \{ \text{dist}(u) + w(u,v) \}$$

Longest increasing Subsequence

5, 2, 8, 6, 3, 6, 9, 7 $\quad s_i$

2, 3, 6

2, 3, 6, 7

$$LIS(j) = 1 + \max_{i<j} \{LIS(i)\}$$

1. Identify the underlying DAG.
   Identify its vertices and edges.
2. Find out the DP formulation
3. Write the DP Algorithm.

Extra: Demonstrate the DAG or the table.

Knapsack problem:

The knapsack can hold at most W

There are n items.

weights: $w_1, w_2, \ldots, w_n$

values: $v_1, v_2, \ldots, v_n$

Q: Most valuable combination that can be fit?

$W = 10$

w/ repetetion
1 & 4(twice) $v=48$

Item	Weight	Value	
1	6	30	5
2	3	14	4.66
3	4	16	4
4	2	9	4.5

wo/ rep. 1 & 3

$v = 46$

$$K(10) = \max\{K(10-4)+16,\ K(10-3)+14,\ K(10-6)+30,\ K(10-2)+9\}$$

What are the subproblems? w/ rep

$K(w)$: Max value of items that can be fit in a knapsack of capacity w.

Goal: Find $K(W)$

$$K(w) = \max_{i : w_i < w} \{ K(w - w_i) + v_i \}$$

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$$K(w) = \max_{i:\, w_i \leq w} \left\{ K(w - w_i + v_i \right\}$$

$$K(w) = \max_{i : w_i < w} K(w - w_i) + v_i$$

$K(0) = 0$

for ($w = 1$ to W)

$$K(w) = \max_{w_i \leq w} \{K(w - w_i) + v_i\}$$

return $K(W)$

$O(nW)$

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$K(\omega, j) =$ max value achived

Goal: $K(W, n)$

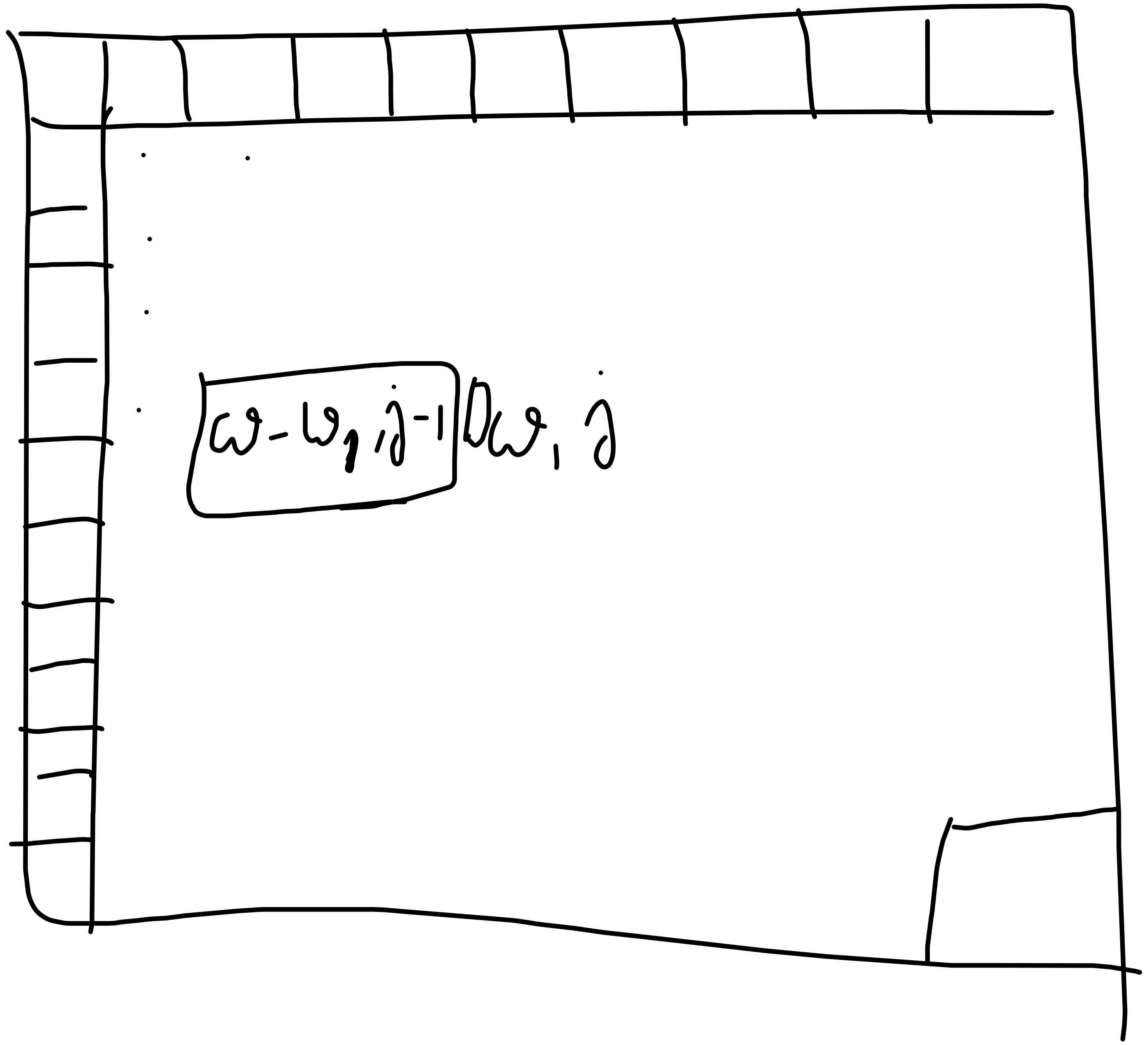
$K(w, j)$ = max value achived in a knapsack of capacity w and items $1, \ldots, j$.

Goal: $K(W, n)$

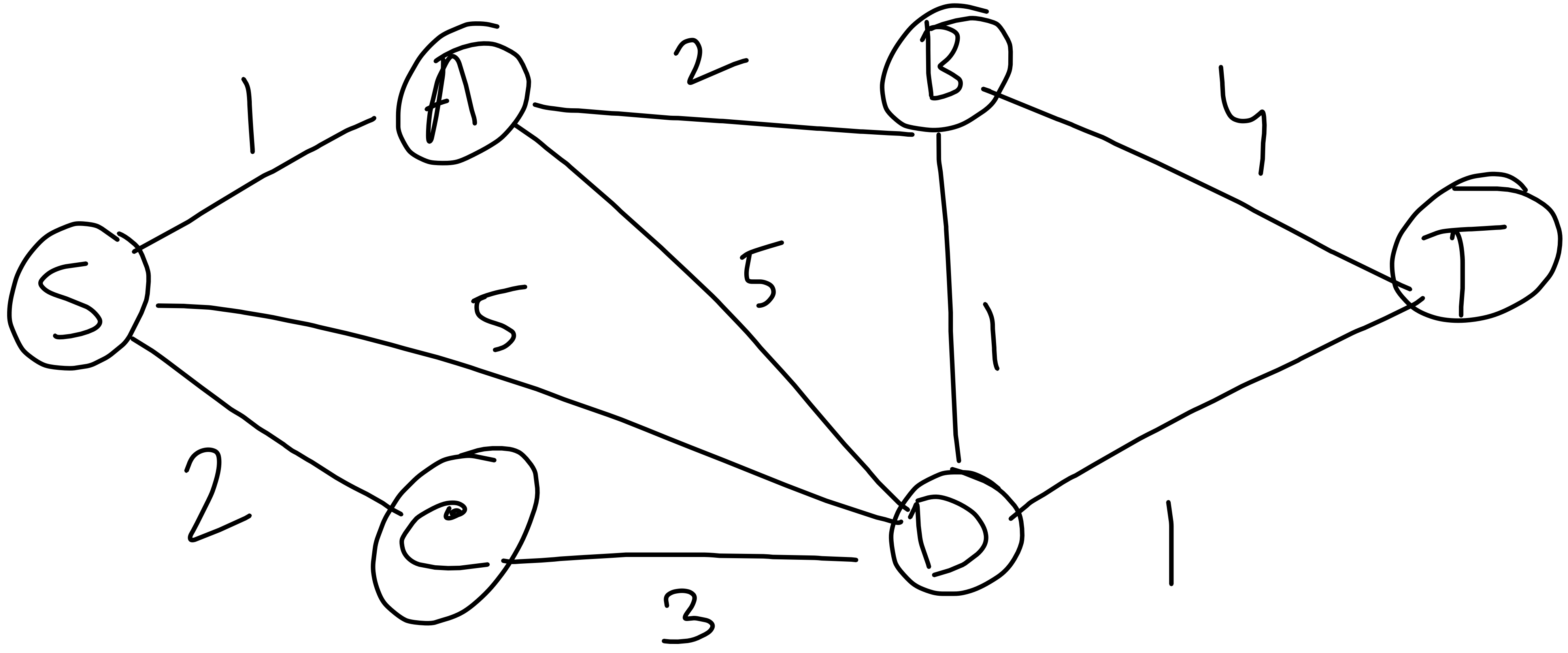
$$K(w, j) = \max \begin{cases} K(w - w_j, j-1) + v_j & \text{if } w \geq w_j \\ K(w, j-1) \end{cases}$$

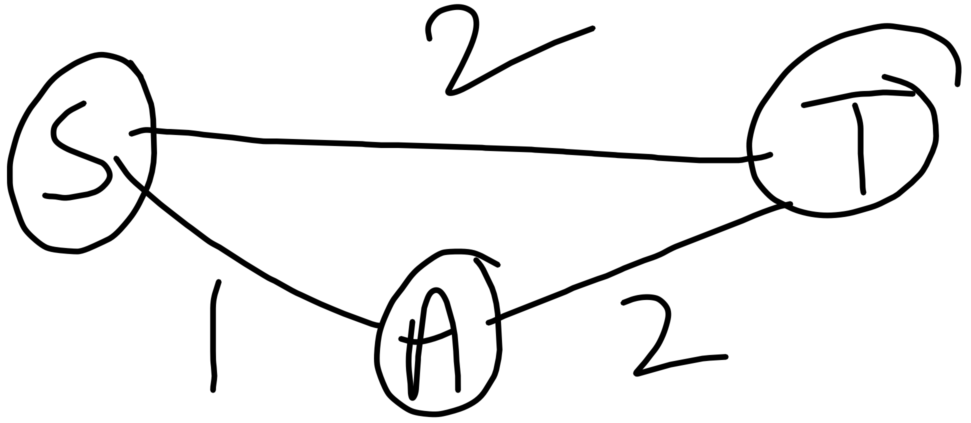
if jth item is included

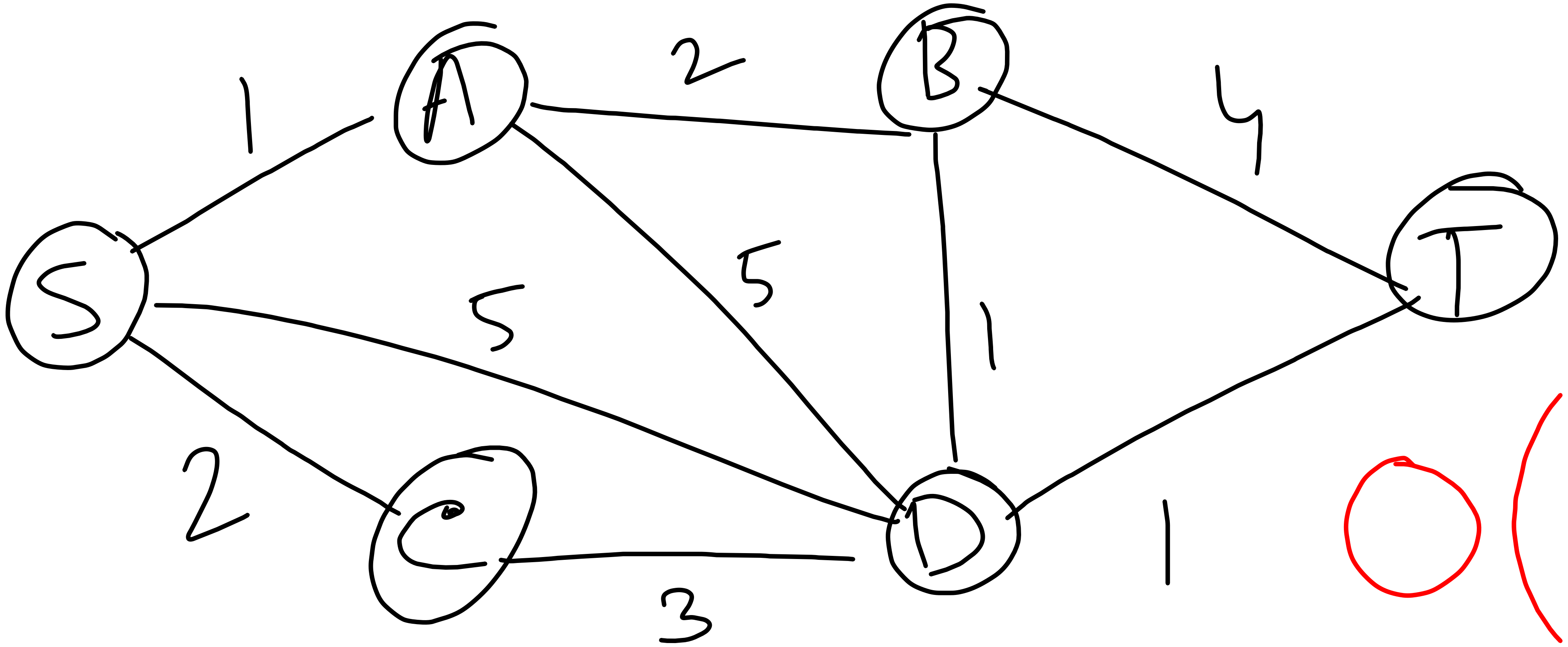
if jth item is not included



$n \times W$


S
A
B
C
D
T
1
2
4
5
5
1
2
3
1


S
A
T
2
1
2



$$O(|V|^2|E|)$$

$$\approx O(|V|^4)$$

Shortest reliable path

Shortest path with least number of edges.

$$\mathrm{dist}(v,i) = \min_{(u,v)\in E} \left\{ \mathrm{dist}(u,i-1) + l(u,v) \right\}$$

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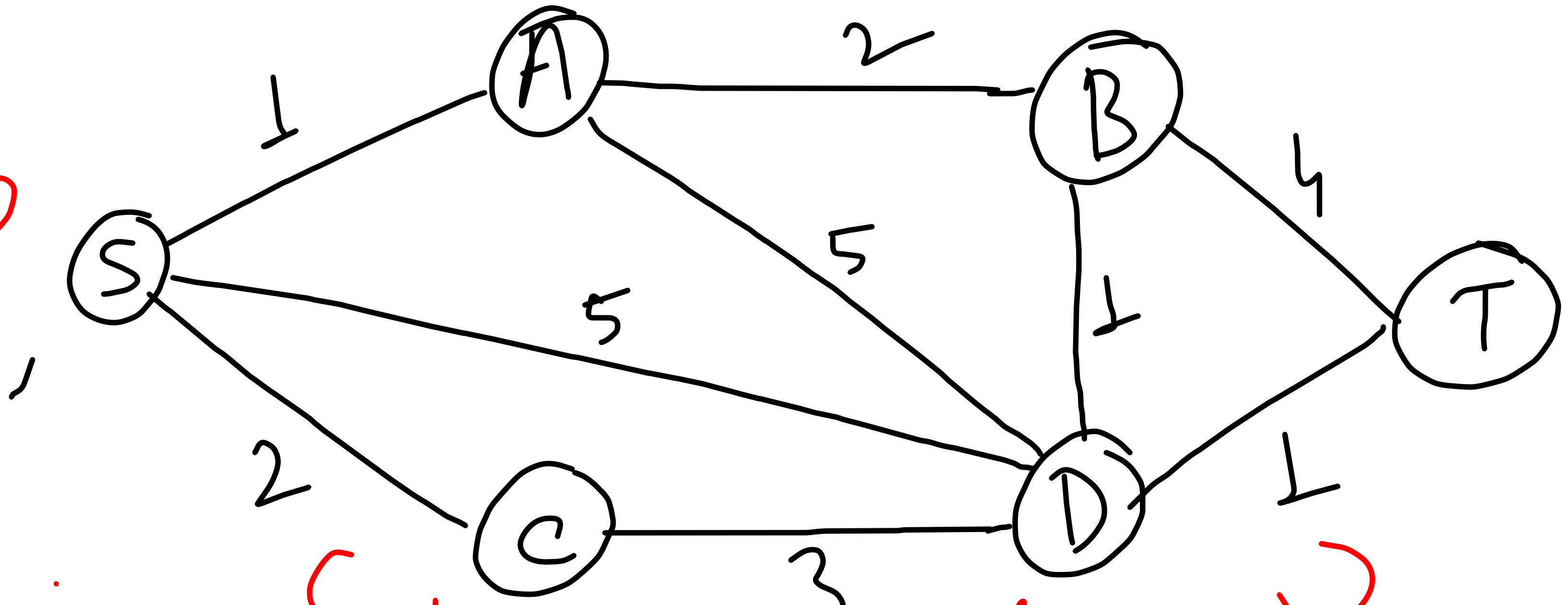
$$dist(v,i) = \min_{(u,v)\in E} \left\{ dist(u,i-1) + l(u,v) \right\}$$

$$dist(s,0) = 0$$
$$dist(v,0) = \infty$$

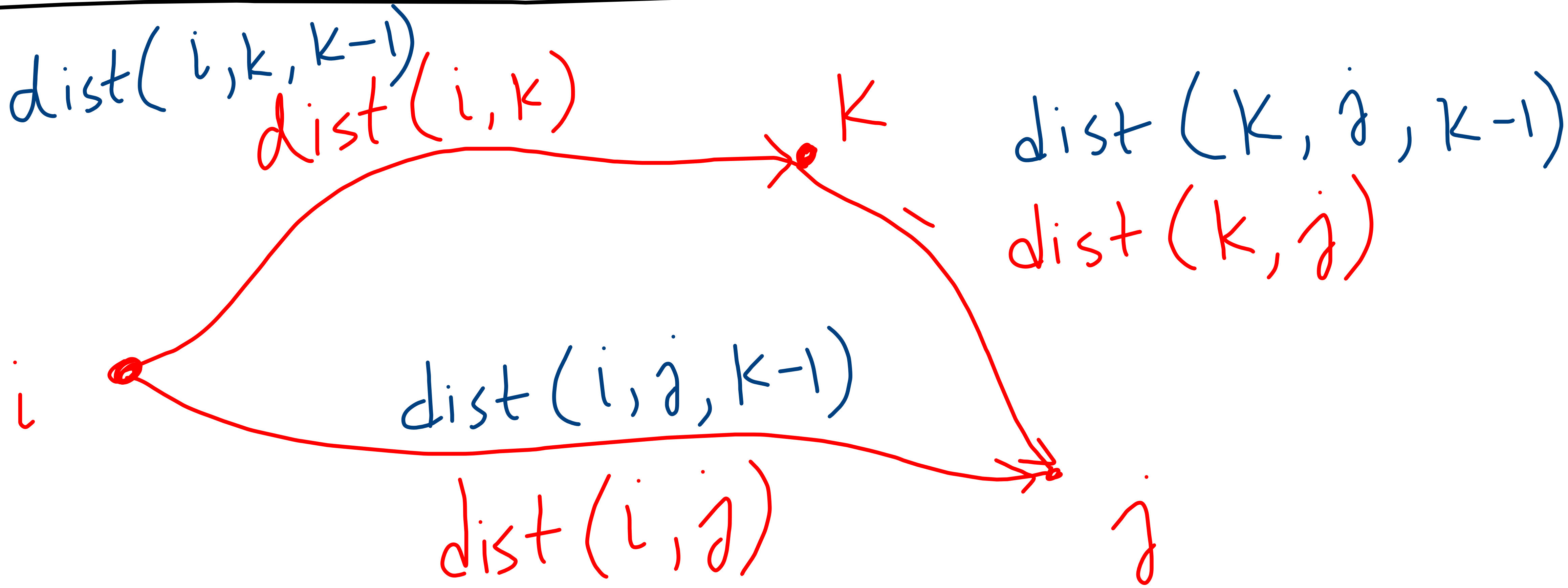


$$dist(v,i) = \min_{(u,v)\in E} \left\{ dist(u,i-1) + \ell(u,v) \right\}$$

	0	1	2	3	4	5
S	0	0	0	0	0	0
A	∞	1(S)	1(S)	1(S)	1(S)	1(S)
B	∞	∞	3(A)	3(A)	3(A)	3(A)
C	∞	2(S)	2(S)	2(S)	2(S)	2(S)
D	∞	5(S)	5(S)	4(B)	4(B)	4(B)
T	∞	∞	6(D)	6(D)	5(D)	5(D)

$$dist(A,2)$$
$$= \min \left\{ \begin{array}{l} dist(S,1) + \ell(S,A), \\ dist(B,1) + \ell(B,A), \\ dist(D,1) + \ell(D,A) \end{array} \right\}$$
$$= \min \{1, \infty, 10\}$$
$$= 1$$

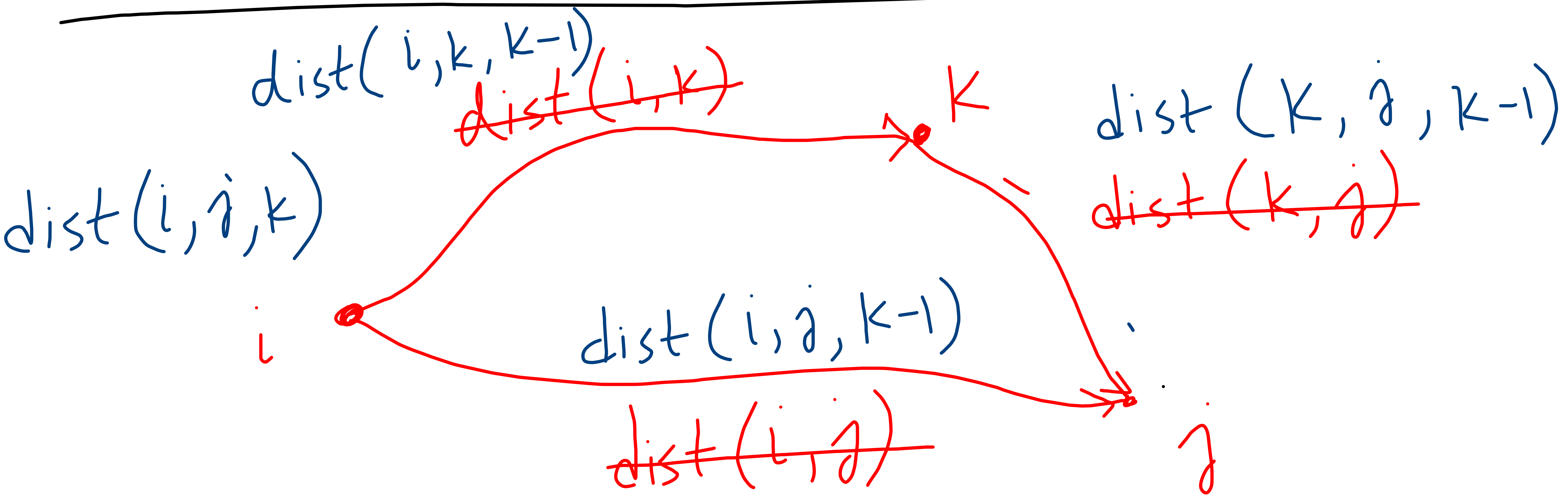
All pairs shortest path



permissible intermediate vertices

Find the shortest path from i to j using vertices from 1 through k.

All pairs shortest path



permissible intermediate vertices

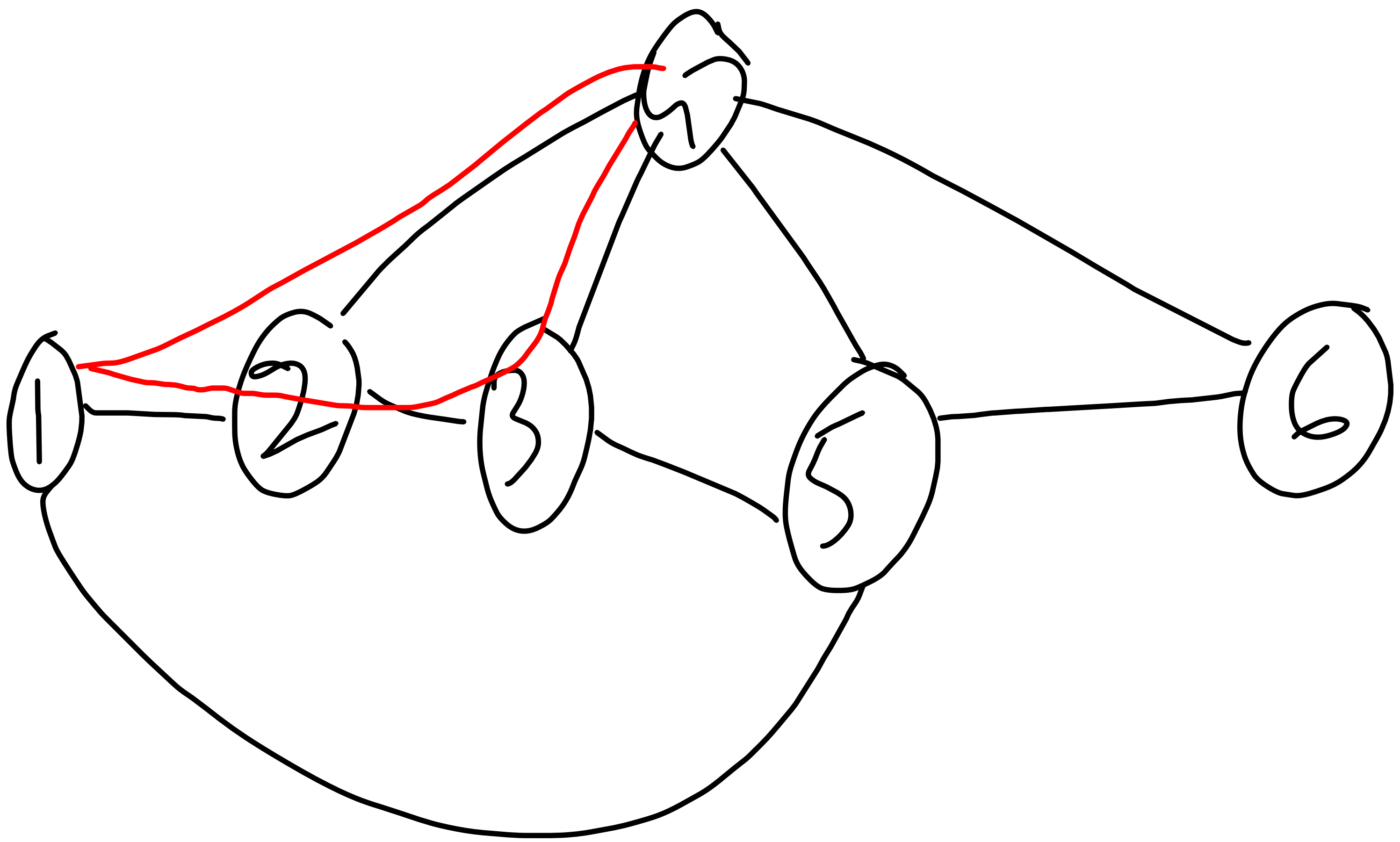
Find the shortest path from i to j using vertices from 1 through k.

$$\mathrm{dist}(i,j,k) = \min\left\{ \mathrm{dist}(i,k,k-1) + \mathrm{dist}(k,j,k-1),\ \mathrm{dist}(i,j,k-1) \right\}$$

$$\mathrm{Ans} = \mathrm{dist}(i,j,n) \qquad O(|V|^3)$$

$$\mathrm{dist}(i,j,0) = \begin{cases} \ell(i,j) & \text{if } (i,j) \in E \\ \infty & \text{if } (i,j) \notin E \\ 0 & \text{if } i = j \end{cases}$$

$$\mathrm{dist}(1,6,4) = \min \left\{ \begin{array}{l} \mathrm{dist}(1,4,3) + \mathrm{dist}(4,6,3), \\ \mathrm{dist}(1,6,3) \end{array} \right\}$$

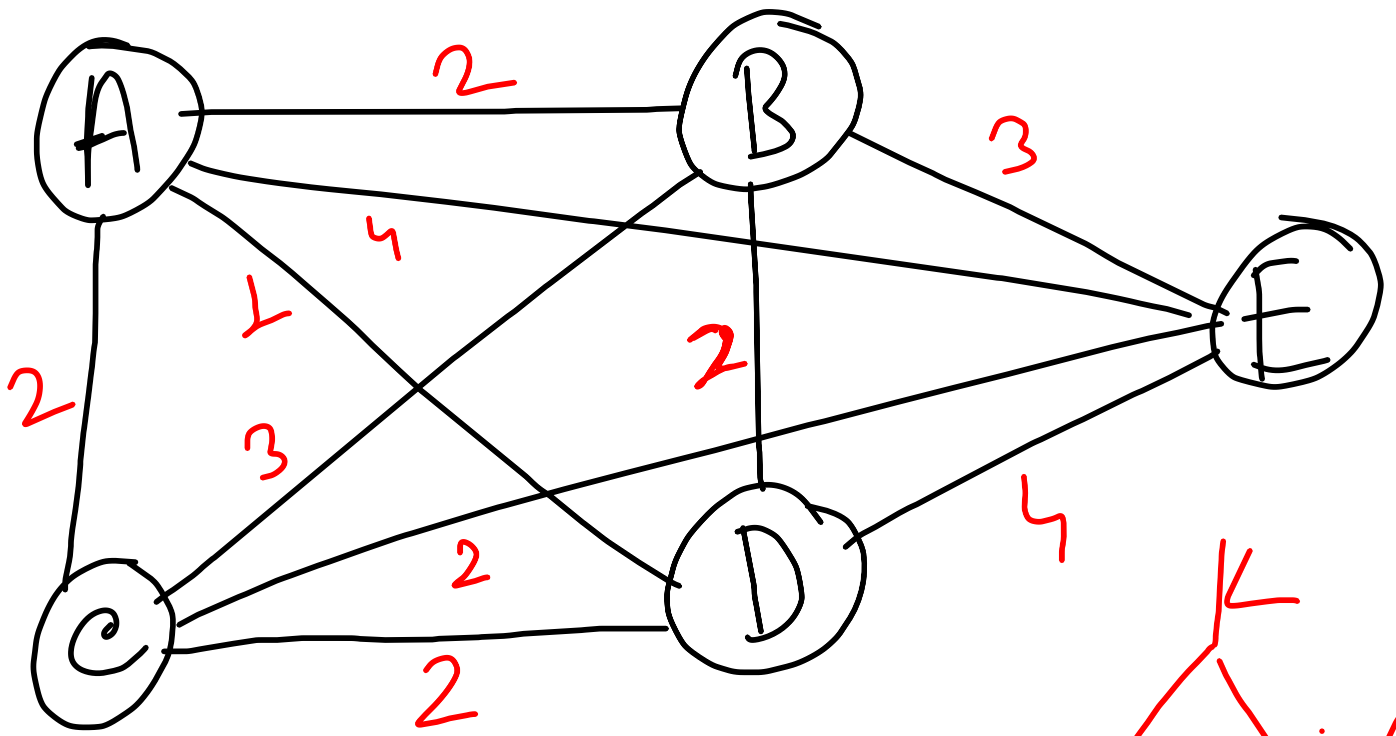


Traveling Salesman problem

$O((n-1)!)$ $\qquad$ $O(n^2 2^n)$

$c(i, k)$: $\qquad$ $c(i, k-1)$

$A \overset{2}{-} B \overset{3}{-} E \overset{2}{-} C \overset{2}{-} D \overset{1}{-} A$



$D = d_{ij}$

$P_1 : A \;-\;-\;-\;-\; v$

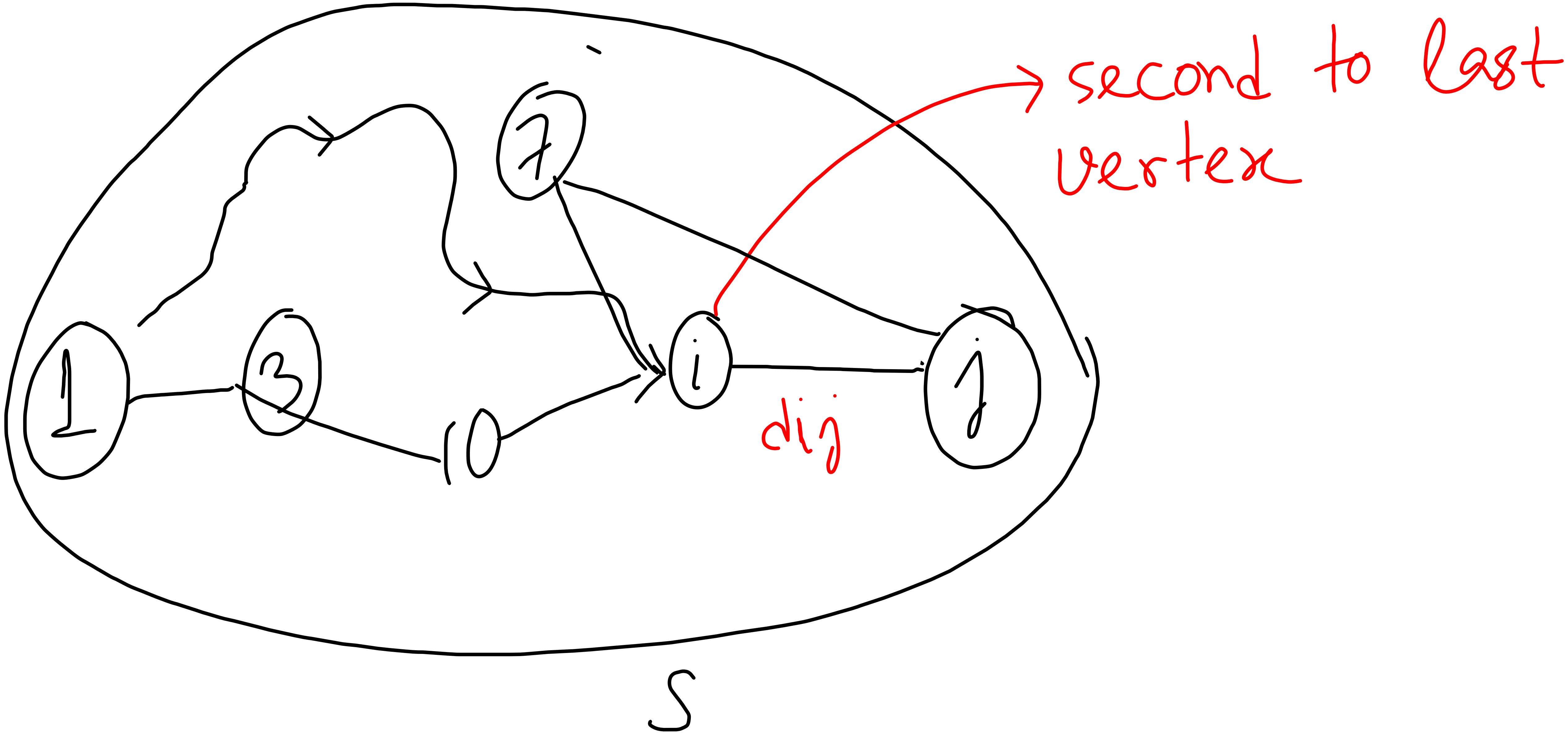
$P_2 : v \;-\;-\;-\; A$

For a subset of vertices $S \subseteq \{1, \ldots, n\}$ that includes 1, and $j \in S$

$C(S, j)$ = length of the shortest path visiting each node in S exactly once, starting at 1 and ending at j.

$|S| > 1$, $C(S, 1) = \infty$

$$c(S, j) = \min_{i \in S : i \neq j} c(S - \{j\}, i) + d_{ij}$$



$c(\{1\}, 1) = 0$ $O(n^2 2^n)$

for $s = 2$ to n $O(n)$

for all subsets $S \subseteq \{1, 2, \dots n\}$ of size s and containing 1 $O(2^n)$

$c(S, 1) = \infty$

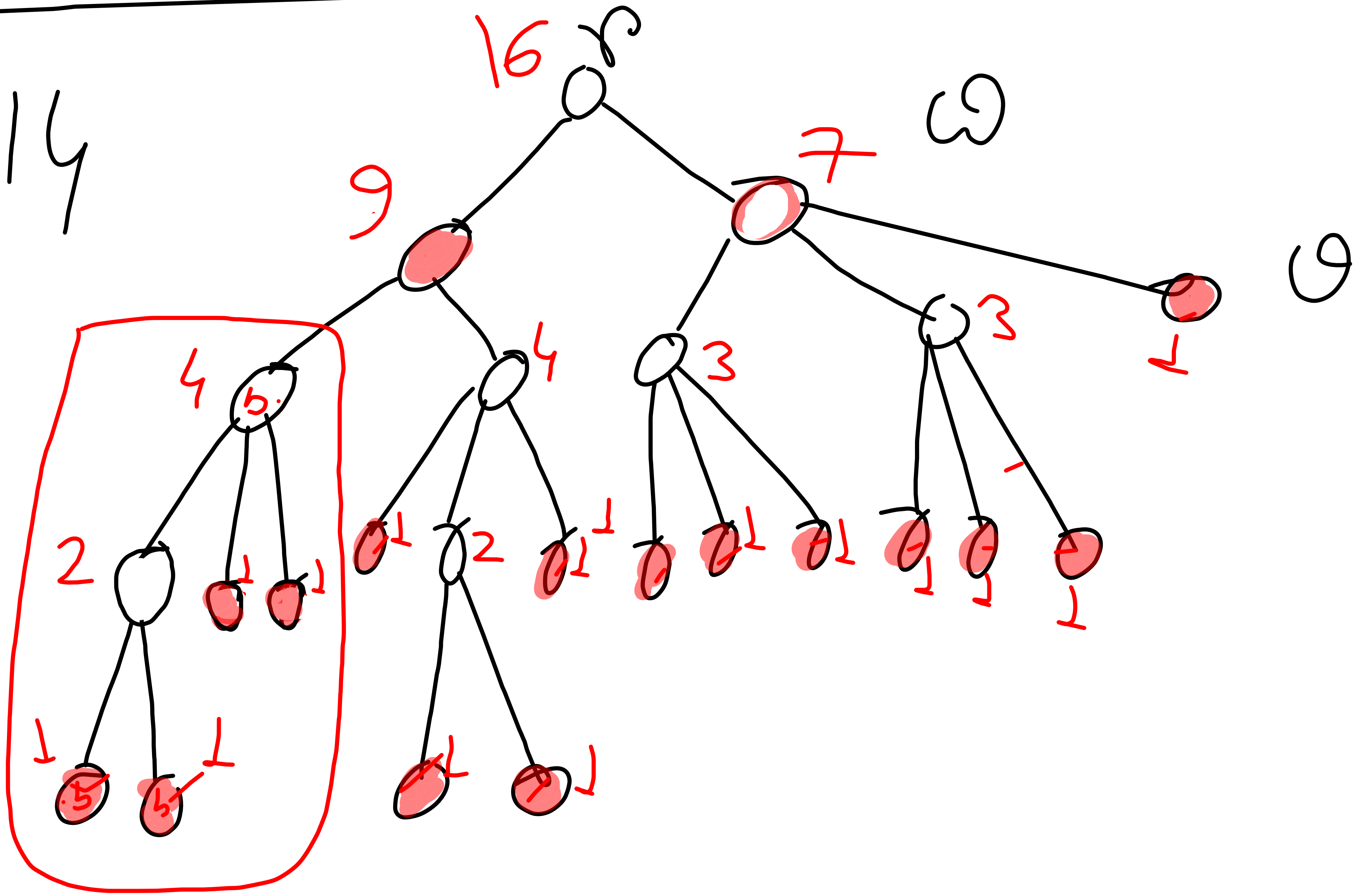
for all $j \in S, j \neq 1$ $O(n)$

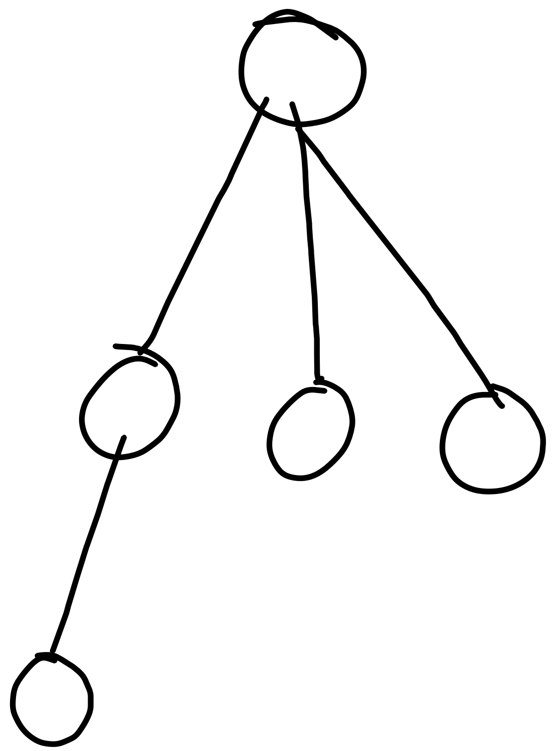
$$c(S, j) = \min_{i \in S, i \neq j} \{ c(S - \{j\}, i) + d_{ij} \}$$

return $\min_j c(\{1, \dots n\}, j) + d_{j,1}$

Max Independend Set in a Tree

$24 = 10 + 14$





$I(w)$: Size of the Max Ind set in the subtree rooted at vertex w.

$$I(w) = \max\left\{ 1 + \sum_{\substack{v \text{ is a} \\ \text{grandchild of } w}} I(v) \;,\; \sum_{\substack{x \text{ is a child} \\ \text{of } w}} I(x) \right\}$$

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Matrix Chain Multiplication

$A_{m\times n} \times B_{n\times p} \times C_{p\times q} \times D_{q\times r}$

$50\times 20 \quad 20\times 1 \quad 1\times 10 \quad 10\times 100$

$A \times ((B \times C) \times D)$	$20\cdot 1\cdot 10 + 20\cdot 10\cdot 100 + 50\cdot 20\cdot 100$	120,200
$((A \times (B \times C)) \times D)$	$20\cdot 1\cdot 10 + 50\cdot 20\cdot 10 + 50\cdot 10\cdot 100$	60,200
$(A \times B) \times (C \times D)$	$50\cdot 20\cdot 1 + 1\times 10\times 100 + 50\cdot 1\times 100$	7,000

$$\overset{1}{d_1},\ \overset{2}{d_2},\ \overset{3}{d_3},\ \overset{2}{\ldots},\ \overset{1}{\ldots},\ d_{n+1}$$

$$\underset{d_1 d_2}{A} \quad \underset{d_2 d_3}{B} \quad \underset{d_3 d_4}{C} \quad \underset{d_4 d_5}{D} \qquad d_3 = \max(d_1, d_2, d_3, d_4, d_5)$$

$$(A \times (B \times C)) \times D \quad \text{or} \quad A \times ((B \times C) \times D)$$

$$d_2 d_3 d_4 + d_1 d_2 d_4 + d_1 d_4 d_5 \ \ 18 \qquad d_2 d_3 d_4 + d_2 d_4 d_5 + d_1 d_2 d_5 \ \ 18$$

$$(A \times B) \times (C \times D) = \underset{6}{d_1 d_2 d_3} + \underset{6}{d_3 d_4 d_5} + \underset{3}{d_1 d_3 d_5} = 15$$