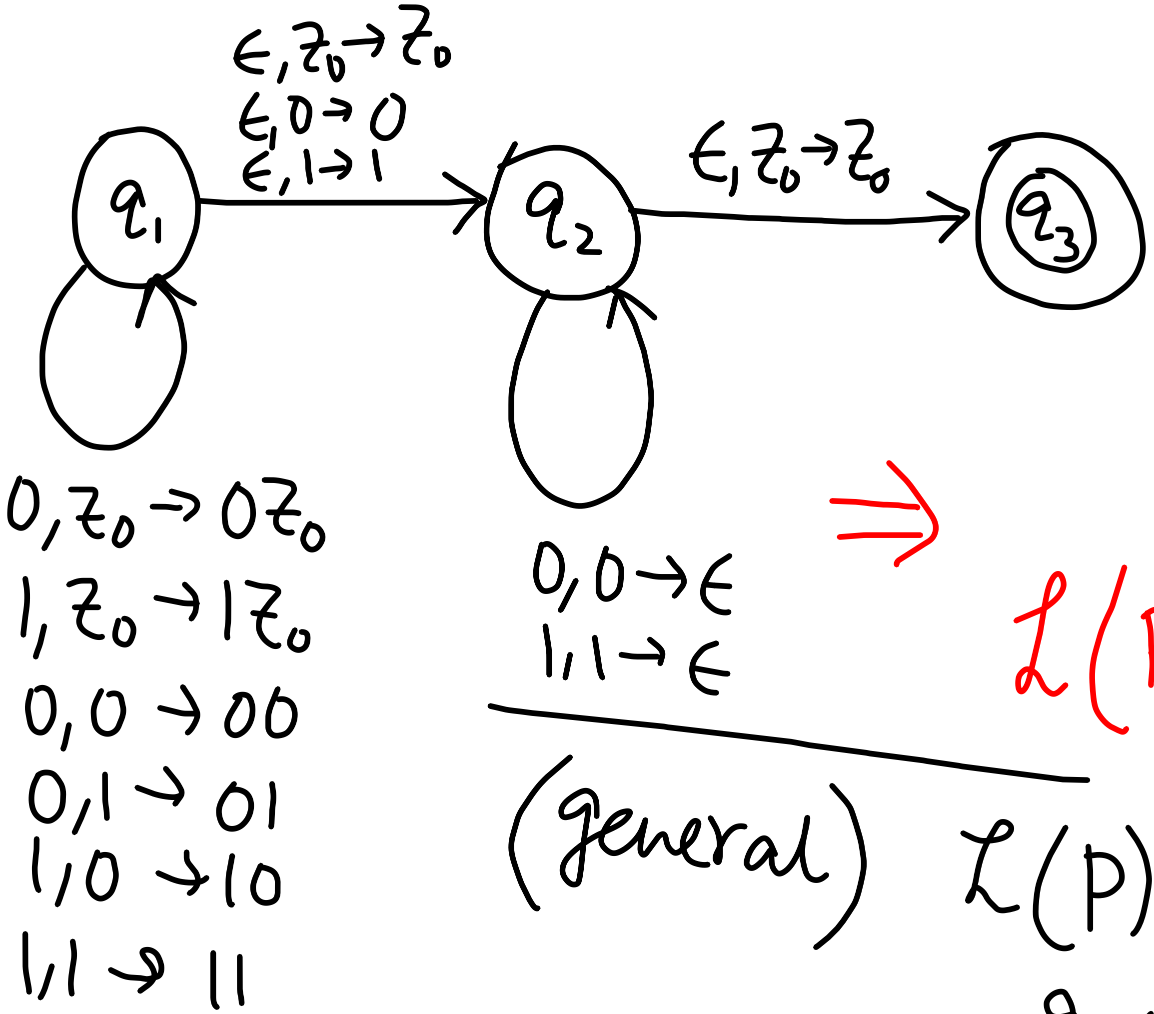
Pushdown Automata (PDA)

PDA for $\mathcal{L} = \{ w w^r \mid w \in \{0,1\}^* \}$

P



Show that the PDA P accepts a string x <u>if and only if</u> $x = w . w^r$ for some ~~x~~ $w \in \{0,1\}^*$

$\Rightarrow$ $\mathcal{L}(P) = \{ w \mid (q_1, w, Z_0) \vdash^* (q_3, \epsilon, Z_0) \}$

(general) $\mathcal{L}(P) = \{ w \mid (q_0, w, Z_0) \vdash^* (q_f, \epsilon, *) \}$

$q_0 \leftarrow$ init state, $q_f \in F$

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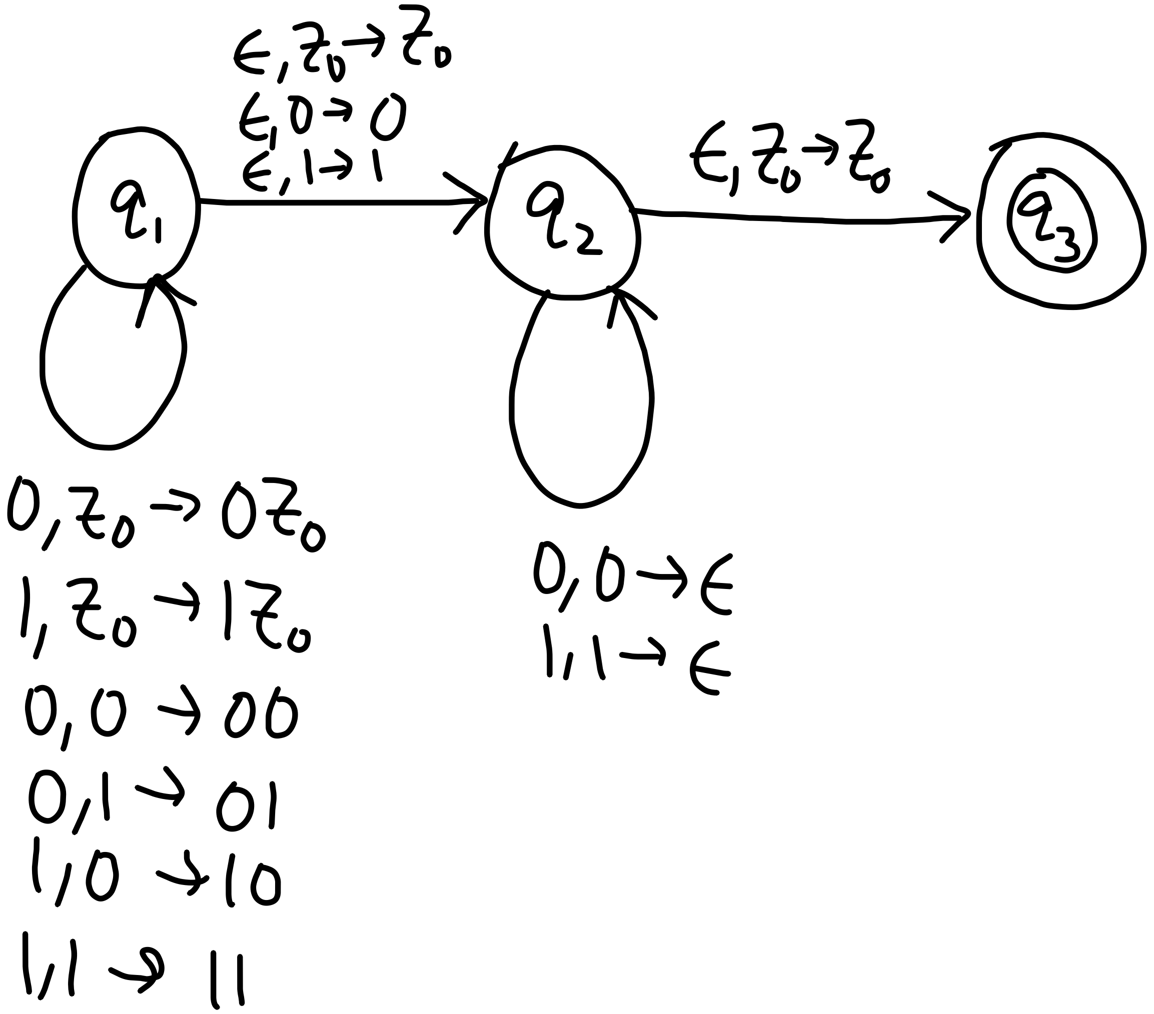
PDA for $L = \{ w w^r \mid w \in \{0,1\}^* \}$

P



Show that the PDA P accepts a string x if and only if $x = w.w^r$ for some $\cancel{x}\, w \in \{0,1\}^*$

Instantaneous Description (ID) of a PDA

$$(q, w, \gamma)$$

- q: Current state
- w: input to be consumed
- γ: Current stack symbols

Define a <u>binary relation</u> "$\vdash$" (turnstyle) on the set of IDs of a PDA P:

$$(q, aw, X\gamma) \vdash (p, w, Y\gamma)$$

$$\text{if } \delta(q, a, X) = \{(p, Y), \ldots\}$$

Example:

$$
\begin{tikzcd}
(q_1, 0110, Z_0) \arrow[d] \arrow[drr] & & & & \\
(q_1, 110, 0Z_0) \arrow[d] \arrow[dr] & & (q_2, 0110, Z_0) \arrow[dr] & & \\
(q_1, 10, 10Z_0) \arrow[d] \arrow[dr] & (q_2, 110, 0Z_0) & & (q_3, 0110, Z_0) & \\
(q_1, 0, 110Z_0) & (q_2, 10, 10Z_0) \arrow[r] & (q_2, 0, 0Z_0) \arrow[r] & (q_2, \epsilon, Z_0) \arrow[r] & \boxed{(q_3, \epsilon, Z_0)}
\end{tikzcd}
$$

Example:

$$
\begin{tikzcd}
(q_1, 0110, Z_0) \arrow[d] \arrow[r] & (q_2, 0110, Z_0) \arrow[r] & (q_3, 0110, Z_0) & & \\
(q_1, 110, 0Z_0) \arrow[d] \arrow[r] & (q_2, 110, 0Z_0) & & & \\
(q_1, 10, 10Z_0) \arrow[d] \arrow[dr] & & & & \\
(q_1, 0, 110Z_0) & (q_2, 10, 10Z_0) \arrow[r] & (q_2, 0, 0Z_0) \arrow[r] & (q_2, \epsilon, Z_0) \arrow[r] & \boxed{(q_3, \epsilon, Z_0)}
\end{tikzcd}
$$

Properties of ID

(P1) If P is a PDA & $(q, x, X) \vdash_P^* (p, y, Y)$, then

for any $w \in \Sigma^*$, $z \in \Gamma^*$,

$$(q, xw, XZ) \vdash_P^* (p, yw, YZ).$$

(P2) $(q, xw, X) \vdash^* (p, yw, Y)$ then,

$$(q, x, X) \vdash_P^* (p, y, Y).$$

Ques

$(q, xw, XZ) \vdash^* (p, yw, YZ)$

then

$(q, x, X) \vdash^* (p, y, Y)$?

(NO)

Correctness

(If) if $x = w.w^r$ then

$$(q_1, w.w^r, Z_0) \vdash^* (q_3, \epsilon, Z_0)$$

Pf:

$$\begin{aligned}(q_1, w.w^r, Z_0) &\vdash^* (q_1, w^r, w^r Z_0)\\ &\vdash (q_2, w^r, w^r Z_0)\\ &\vdash^* (q_2, \epsilon, Z_0)\\ &\vdash (q_3, \epsilon, Z_0)\end{aligned}$$

(Only If)

$(q_p, x, \alpha) \vdash^* (q_3, \epsilon, \alpha)$ then

$x = w \cdot w^r$

[We prove for general α. Here $\alpha = Z_0$]

$\boxed{x = a_1 a_2 \cdots a_n}$

Induction on $|x|$

Base Case: $x = \epsilon \Rightarrow$ trivial

Induction: Case 1: $(q_1, x, \alpha) \vdash (q_2, x, \alpha)$ is the first transition.

$- \ (q_2, x, \alpha) \vdash^* (q_3, \epsilon, \alpha)$ [Not possible as in q_2 each transition is only pop]

Case 2: $(q_1, a_1 \cdots a_n, \alpha)$

$$\left.\begin{aligned} &\vdash (q_1, a_2 \cdots a_n, a_1\alpha) \\ &\vdash^* (q_2, a_n, a_1\alpha) \\ &\vdash (q_2, \epsilon, \alpha) \\ &\vdash (q_3, \epsilon, \alpha) \end{aligned}\right\} \text{Only valid sequence}$$

$\boxed{a_1 = a_n} \Longleftarrow$

$$\Rightarrow (q_1, a_2 a_3 \cdots \underline{\underline{a_n}}, \underline{\underline{a_1 \alpha}}) \vdash^* (q_2, \underline{\underline{a_n}}, \underline{\underline{a_1 \alpha}})$$

$$(P2) \Rightarrow \quad (q_1, \underbrace{a_2 \cdots\cdots, a_{n-1}}_{y \cdot y^r}, a_1 \alpha) \vdash^* (q_2, \epsilon, a_1 \alpha)$$

$$\begin{aligned} x &= a_1 a_2 \cdots a_{n-1} a_n \\ &= a_1 y \cdot y^r \cdot a_1 \\ &= w \cdot w^r, \quad w = a_1 \cdot y \end{aligned}$$

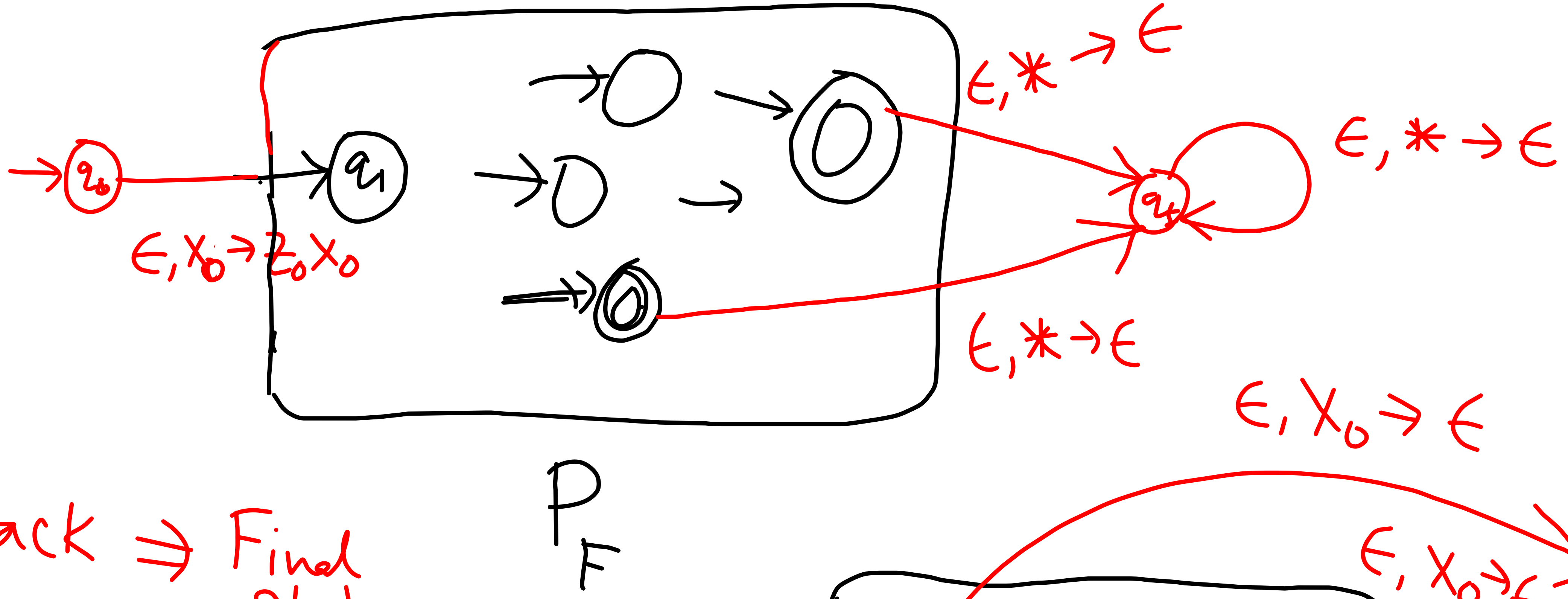
- Acceptance through Final State

Acceptance through Empty Stack

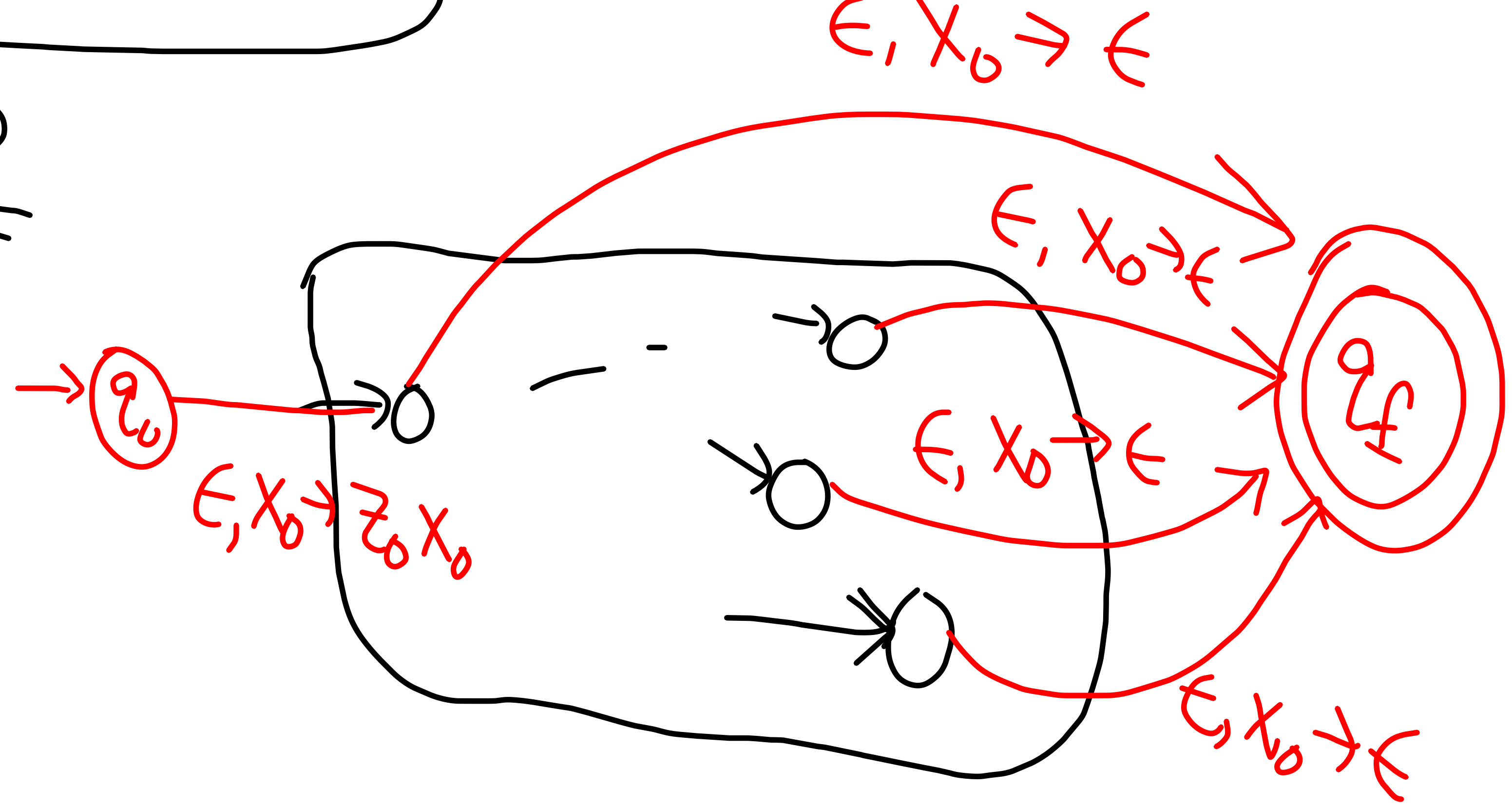
P → PDA

$$N(P) = \{ w \mid \underset{\substack{\downarrow \\ \text{init State}}}{(q_1, w, Z_0)} \vdash^{*} (\underset{\substack{\uparrow \\ \text{any State}}}{q}, \epsilon, \epsilon) \}$$

Final State $\Longrightarrow$ Empty Stack



Empty Stack $\Rightarrow$ Final State



Pallindrome

↳ Single state PDA
with empty Stack

$w \in \{0,1\}^*$

- ww^r
- $w0w^r$
- $w1w^r$

CFG $\Longleftrightarrow$ PDA with empty stack

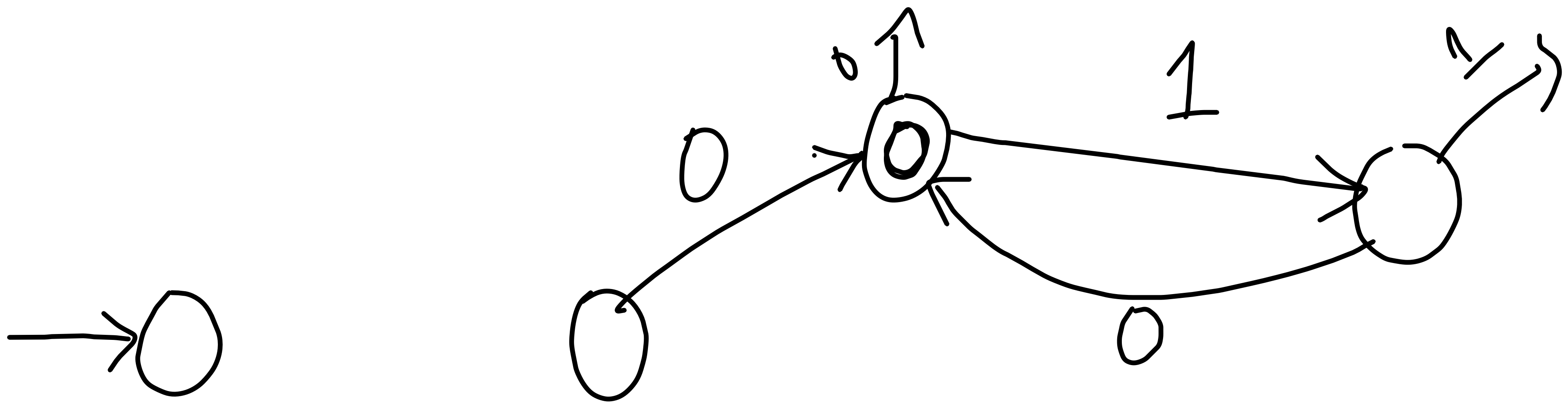
Pallindrome

Single state PDA with empty stack

CFG $\Longleftrightarrow$ PDA with empty stack

$w \in \{0,1\}^*$

- ww^r
- $w0w^r$
- $w1w^r$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