Incidence Geometry

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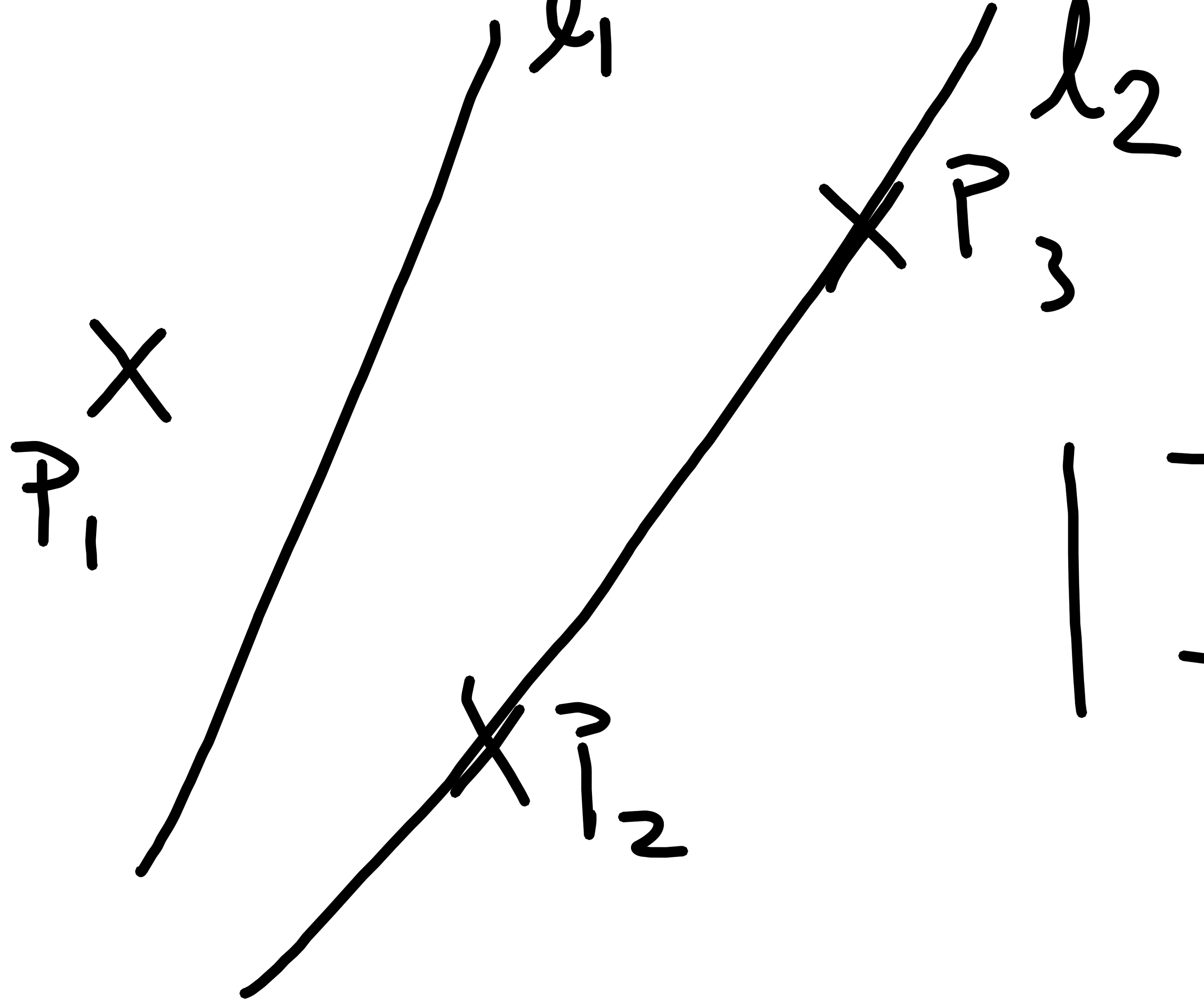
$L^{=m}$ $P^{=n}$ in $\mathbb{R}^2$

$$I(L, P) = \{ (\ell, p) \in L \times P \mid p \in \ell \}$$



$$\{ (P_2, \ell_2), (P_3, \ell_2) \}$$

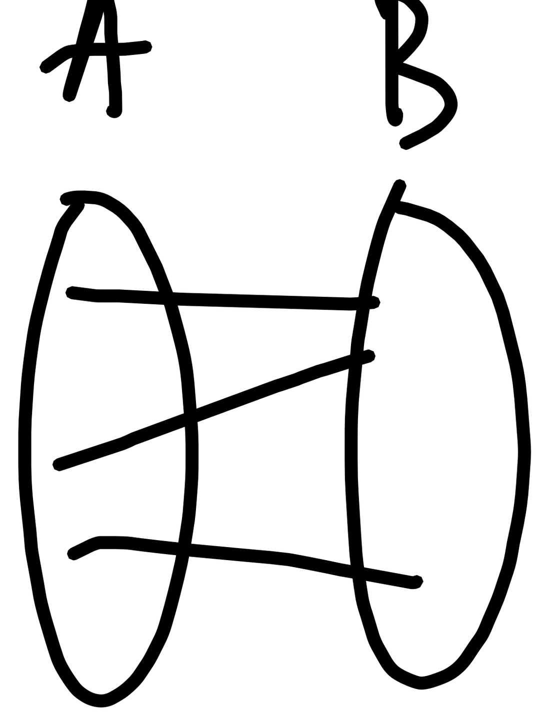
$$|I(P, L)| \lesssim mn$$

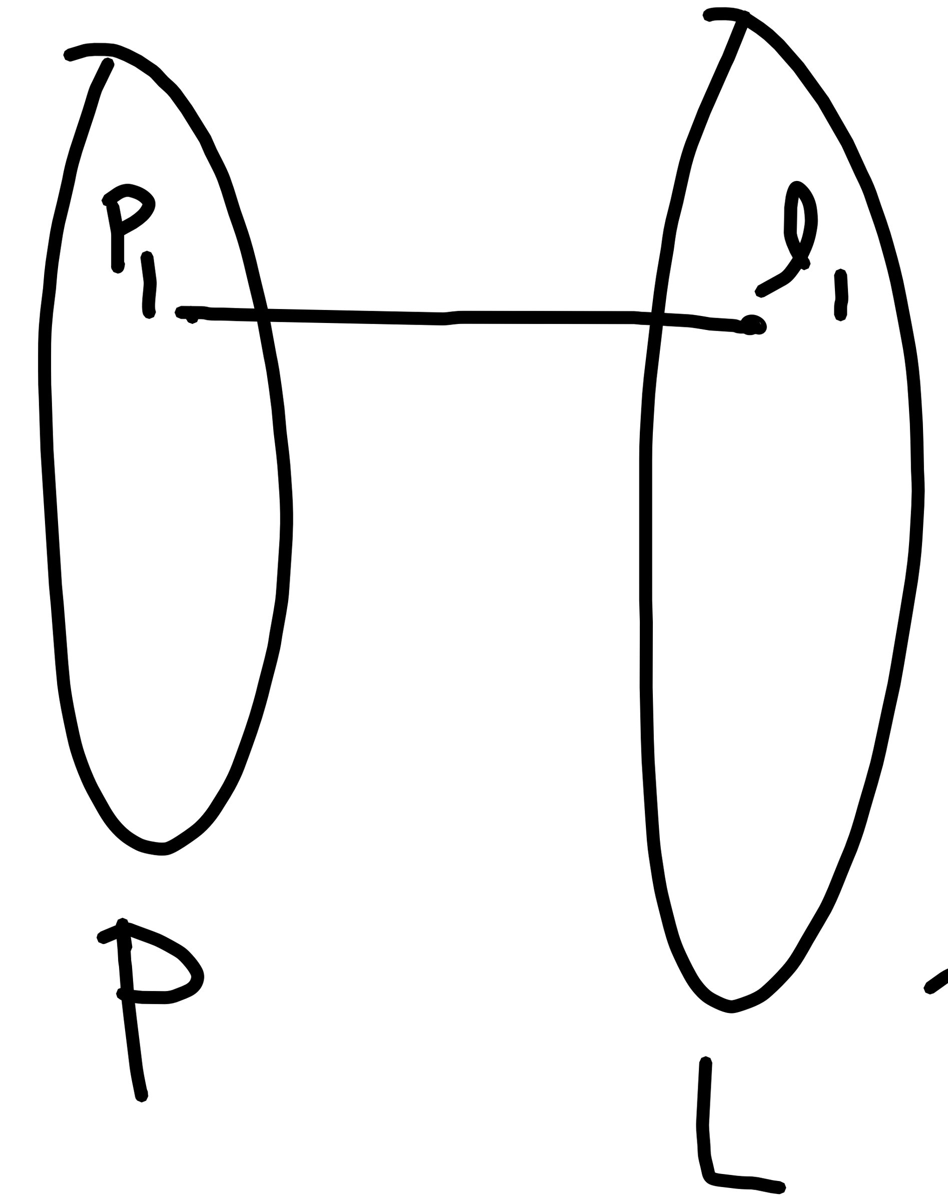
Special case $m = n$

$\mathbb{F}_p^2$

Lemma 1 $|I(P,L)| \lesssim n^{3/2}$

Thm 2 (Kovari-Sos-Turan). $K_{2,2}$-free $|E| \lesssim n^{3/2}$.





Thm 3 (Szemeredi-Trotter '83) P, L lines in $\mathbb{R}^2$

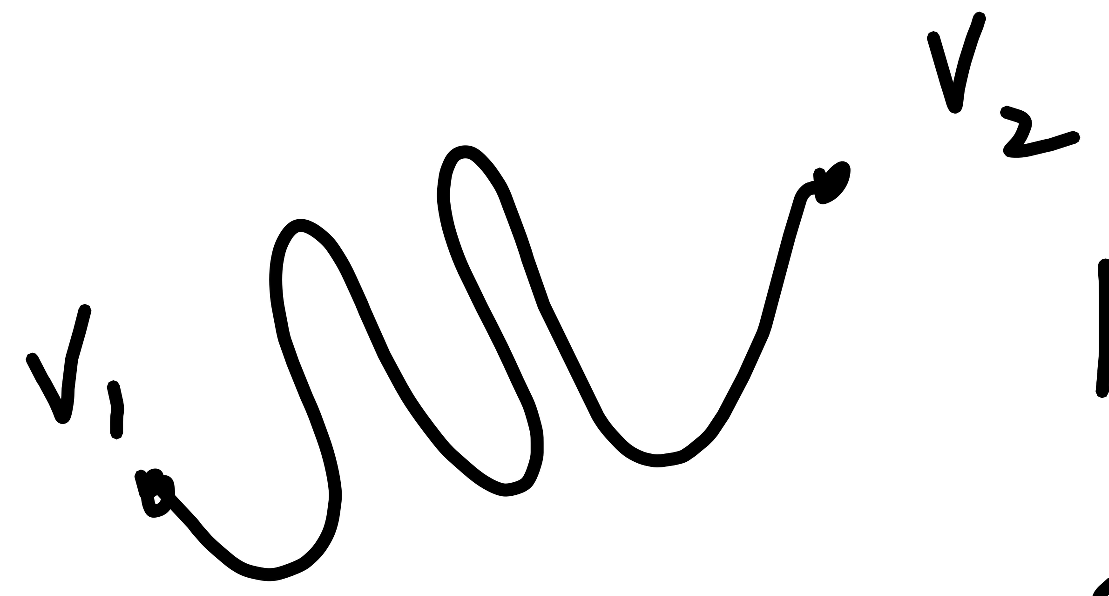
$$|I(P,L)| \lesssim (mn)^{2/3} + m + n$$

Thm 4 (Eq statement) $P \subseteq \mathbb{R}^2$, $|P| = n$

$$|L_k| \lesssim \frac{n^2}{k^3} + \frac{n}{k}$$

$m \geq 10n$

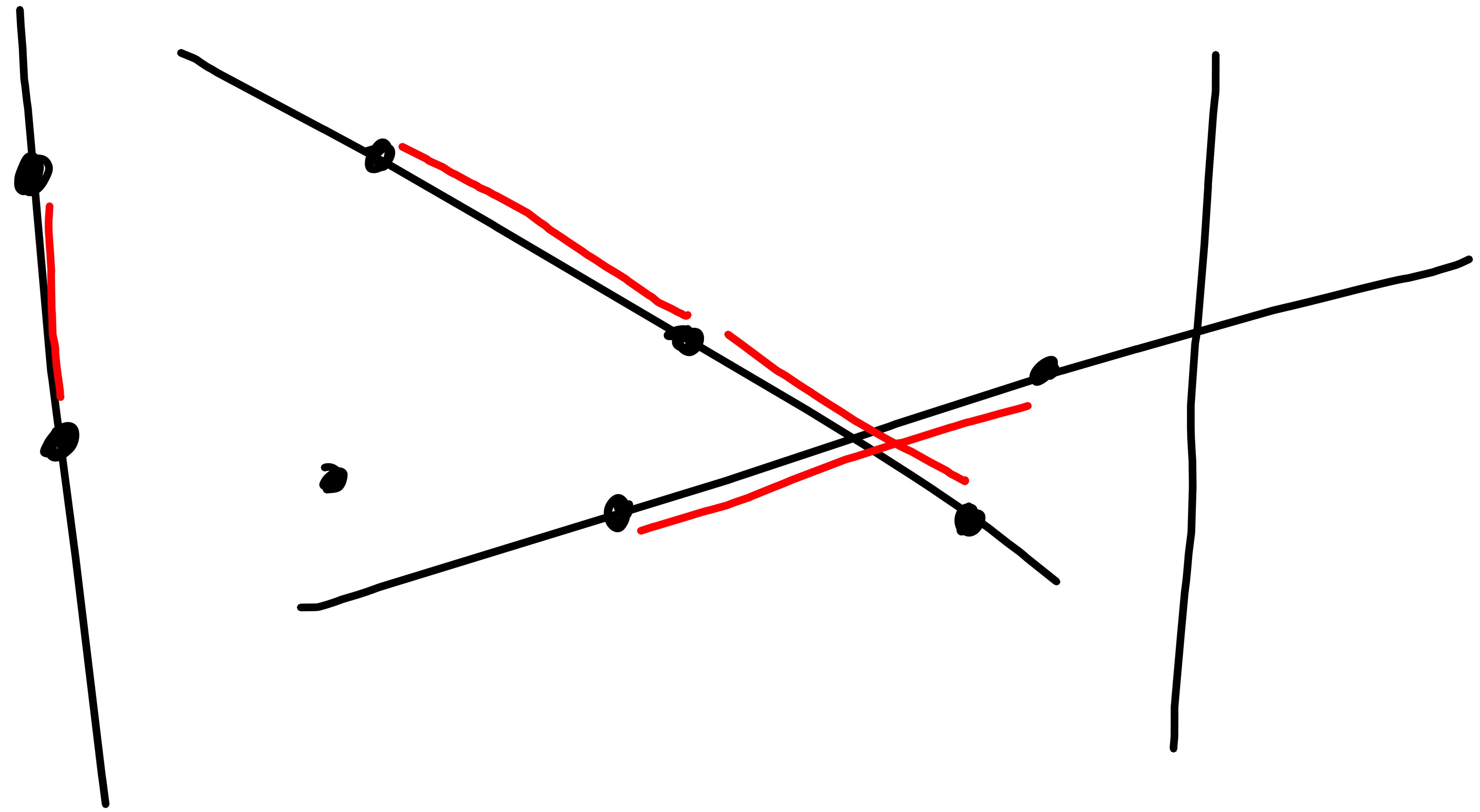
Thm 5 (Crossing lemma) $G = (V, E)$, assume that $|E| \geq 10|V|$



No. of crossings in any drawing of G in $\mathbb{R}^2$ will have at least

$$\gtrsim \frac{|E|^3}{|V|^2}$$

Székely '90s



Applications

(a) $A \subseteq \mathbb{R}$

$$A+A := \{\bar{x}+\bar{y} \mid \bar{x},\bar{y} \in A\}$$
$$A\cdot A = \{\bar{x}\bar{y} \mid \bar{x},\bar{y} \in A\}$$

Erdos-Szemeredi $\max\{|A+A|, |A\cdot A|\} \gtrsim |A|^{1+\epsilon}$ **Conj** $|A|^{2-o(1)}$

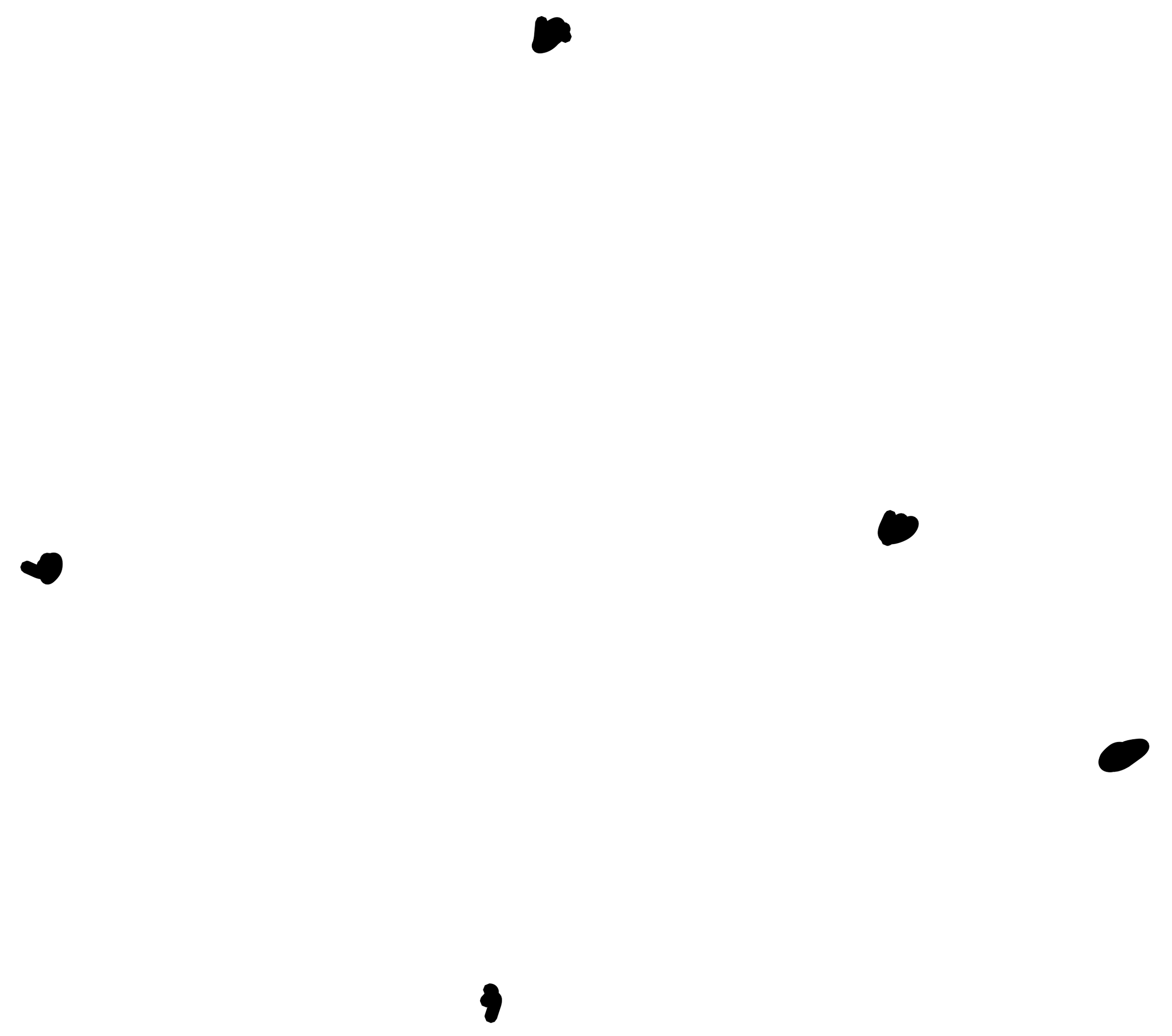
Elekes '97 $\varepsilon \geq 1/4$

Solymosi '05 $\varepsilon \geq 3/11$

Thm 6 (Bourgain-Katz-Tao) $p^{\delta} \leq |A| \leq p^{1-\delta}$

then $\max\{|A+A|, |A\cdot A|\} \to |A|^{1+\varepsilon(\delta)}$

$\mathbb{R}^2$



$Q \in \mathbb{R}[X,Y]$

$\bar{P} = (\bar{x}, \bar{y})$

$Q(\bar{x}, \bar{y}) = 0$

$\exists\, Q \in \mathbb{R}[X,Y]$ with $\deg(Q) \lesssim \sqrt{n}$ s.t. Q covers P.

$n^{1/d}$

$\{0,1\}^n \hookrightarrow \mathbb{R}^n$

Alon-Furedi $\{0,1\}^n \setminus 0^n$

$x_1 = 1$ | $x_i = 1, \forall i \in [n] = \{1, \ldots, n\}$

$x_1 = 0$ | $\sum_{i=1}^{n} x_i = j, \; j = 1, \ldots, n$

Thm (Alon-Furedi) min deg poly covering exactly $\{0,1\}^n \setminus 0^n$ is "n".

Thm 8 (Comb. Nulls) $Q \in \mathbb{F}[x_1, \ldots, x_n]$ with deg d. Also $\exists$ a monomial $x_1^{t_1} \cdots x_n^{t_n}$ with $\sum^n t_i = d$ s.t this monomial is present in Q. Let $S_i \subseteq \mathbb{F}^{i=1}$ s.t $|S_i| > t_i$, then $\exists$ $\bar{x} \in S_1 \times S_2 \cdots \times S_n$ s.t $Q(\bar{x}) \neq 0$.

$\alpha_1 \leftarrow x_1^{t_1} \cdots x_n^{t_n}$ s.t. $\sum t_i = \deg(\alpha_1)$

$$\underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n}$$

$S \subsetneq \{0,1\}^n$

Cor( ) If S is "symmetric set" then min no. hyp = min deg of poly.

$$S = S_1 \times S_2 \times \cdots \times S_t \qquad \sum l_i = n$$

$S_1 \subseteq \{0,1\}^{l_1}$, $S_t \subseteq \{0,1\}^{l_t}$