Sample Question (IAI Ph.D. Admission: 2024)

All the questions of each group carry equal marks.

Group A

Basic Algebra

1. (a) Consider the polynomial $x^5+ax^4+bx^3+cx^2+dx+4$ where a, b, c, d are real numbers. If $(1+2i)$ and $(3-2i)$ are two roots of this polynomial then determine the value of a.

   (b) Determine the number of real roots of the equation

$$2\cos\left(\frac{x^2+x}{6}\right) = 2^x + 2^{-x}.$$

   (c) If $(1+x)^n = C_0+C_1x+C_2x^2+\ldots+C_nx^n$, n being a positive integer, then determine the value of

$$\left(1+\frac{C_0}{C_1}\right)\left(1+\frac{C_1}{C_2}\right)\cdots\left(1+\frac{C_{n-1}}{C_n}\right).$$

2. (a) Determine the number of solutions in $1 \leq x \leq 315$ to the following system of congruent equations:

$$x \equiv 2 \mod 15,\ x \equiv 1 \mod 21.$$

   (b) Find a basis for the following subspace of $\mathbb{R}^4$:

$$\{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + 3x_4 = 0, x_1 + x_2 - 2x_3 + x_4 = 0\}.$$

   Argue whether it is possible to compute the dimension of the above subspace without explicitly finding a basis? Justify your answer.

   (c) Consider the following system of equations over a field $\mathbf{F}$.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2, \end{aligned}$$

   where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbf{F}$. State the conditions for which the above system of equations has (i) no solution, (ii) a unique solution, and (iii) more than one solution.

3. (a) Let G be the group $\{\pm 1, \pm i\}$ with multiplication of complex numbers as composition. Let H be the quotient group $\mathbb{Z}/4\mathbb{Z}$. Then determine the number of non-trivial group homomorphisms from H to G.

(b) Let G be the set of all real 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $a \neq 0$ and $d \neq 0$. (i) Show that$G$ forms a group under matrix multiplication. (ii) Show that $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$ is a normal subgroup of G. (iii) Show that G/H is an abelian group.

4. (a) Let S_n be the group of all permutations of $\{1, \ldots, n\}$ under composition. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

be an element of S_6. (i) Find the order of the cyclic subgroup generated by σ. (ii) Find the minimum n such that S_n contains a cyclic subgroup of order 30. Justify your answer. (iii) Let S be a cyclic group of order 6. Show that S has a unique subgroup of order 3. (iv) Let S be a finite cyclic group and K be a subgroup of S oforder m. Show that an element $a \in S$ is an element of K ifand only if $a^m = e$.

(b) Let G be the group of non-zero complex numbers under multiplication and let N be the set of complex numbers of absolute value 1 (i.e., $a + bi \in N$ if $a^2 + b^2 = 1$). Show that G/N is isomorphic to the group of all positive real numbers under multiplication.

5. (a) Let G be a group such that $(ab)^p = a^p b^p$ for all $a, b \in G$, where p is a prime number. Let

$$S = \{x \in G : x^{p^m} = e \text{ for some } m \text{ depending on } x\}.$$

Prove that S is a normal subgroup of G. If $\overline{G} = G/S$ and $\overline{x} \in \overline{G}$ is such that $\overline{x}^p = \overline{e}$, then $\overline{x} = \overline{e}$.

(b) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are integers modulo p, p is a prime such that $ad - bc \neq 0$. Prove that G forms a group under matrix multiplication. Assuming $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid ad - bc = 1 \right\}$, find $o(H)$.

Elementary Linear Algebra

1. (a) Let $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalue of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}$$

If $\lambda_1 + \lambda_2 + \lambda_3 = \sqrt{2} + 1$, then determine the set of values of t.

(b) Let $\Theta = 2\pi/67$. Consider the following matrix

$$A = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}$$

Then determine the matrix A^{100}.

2. (a) Let A and B be two $n \times n$ symmetric matrices such that $AB = BA$. Show that if $x \neq 0$ is an eigenvector of A and $Bx \neq 0$, then Bx is also an eigenvector of A corresponding to the same eigenvalue.

   (b) Let B be a non-singular matrix. Prove that λ is an eigenvalue of B if and only if λ^{-1} is an eigenvalue of B^{-1}.

3. (a) If $\text{rank}(A) = \text{rank}(A^2)$, then show that $\{x : Ax = 0\} = \{x : A^2x = 0\}$.

   (b) Let A be an $n \times n$ symmetric matrix and let $l_1, l_2, \ldots, l_{r+s}$ be $(r+s)$ linearly independent $n \times 1$ vectors such that for all $n \times 1$ vectors x, $x^{\mathsf{T}}Ax = (l_1^{\mathsf{T}}x)^2 + \ldots + (l_r^{\mathsf{T}}x)^2 - (l_{r+1}^{\mathsf{T}}x)^2 - \ldots - (l_{r+s}^{\mathsf{T}}x)^2$. Prove that $\text{rank}(A) = r + s$.

4. (a) Let V denote the vector space $\mathbb{R}$. Suppose $V \to \mathbb{R}^n$ is a function satisfying

   - $f(v_1, \ldots, v_n) = 0$ whenever $v_i = v_j$ for some $i \neq j$
   - $f(v_1, \ldots, v_{i-1}, \alpha v_i, v_{i+1}, \ldots, v_n) = \alpha f(v_1, \ldots, v_n)$
   - $f(v_1, \ldots, v_{i-1}, v_i + u_i, v_{i+1}, \ldots, v_n) = f(v_1, \ldots, v_n) +$
     $f(v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_n)$
   - $f(e_1, \ldots, e_n) = 1$ where e_i is i-th unit vector,

   where $\alpha \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$. Show that for any $n \times n$ matrix A, whose columns are $v_1, \ldots, v_n$, $f(v_1, \ldots, v_n) = \det(A)$.

   (b) If T is an injective homomorphism of a finite dimensional vector space V onto a vector space W, prove that T maps a basis of V onto a basis of W.

Basic Statistics

1. (a) Let $f(k) = \frac{1}{k(K+1)}$ for integers $k \geq 1$. Show that f is a probability mass function and find the corresponding cumulative density function. What are its mean and median?

   (b) A coin (need not be fair) is tossed till the kth head shows up. Let X be the number of tosses required. Find the probability mass function of X.

2. (a) Find the mode of $Gamma(\alpha, 1)$ density and the mode of $Poi(\lambda)$ distribution.

(b) Throw r distinguishable balls into m labelled bins, independently and with probabilities $p_1, \ldots, p_m$ ($p_1 + \ldots + p_m = 1$). Let X be the number of empty bins. Find the expectation and variance of X.

(c) For a random variable X, show that :

i. $\mathbb{E}[(X-a)^2]$ is minimized when $a = \mathbb{E}[X]$.

ii. $\mathbb{E}[|X-a|]$ is minimized at the medians of X.

3. (a) A box contains p balls carrying label 1, another p balls carrying label 2 and so on up to p coupons labelled N. Draw two balls uniformly at random without replacement and note their labels as X and Y. Find the mean and variance of X and of Y and the covariance of X and Y. What would be the co-variance, when $p \to \infty$?

(b) From a box of N balls labelled $1, 2, \ldots, N$, we draw three balls one after another, uniformly at random. Let X_1, X_2, X_3 be the numbers on the balls. Find the mean and variance of $X_1 + X_2 + X_3$ in the following two situations: (i) the balls are drawn with replacement, (ii) the balls are drawn without replacement.

4. (a) For $X \sim Exp(\lambda)$, find the moments of X. Also find the mean and variance of $\log X$. Can you justify (without calculations) why the variance of $\log X$ should be independent of λ?

(b) For independent random variables X and Y, find the distribution of $Z = T(X, Y)$ for

i. X, Y are i.i.d $N(0,1)$ and $T(x, y) = x/y$.

ii. X, Y are i.i.d $Unif[0,1]$ and $T(x, y) = x + y$.

5. (a) If X and Y are Bernoulli random variables, then show that X and Y independent if and only if they are uncorrelated.

(b) Let U be a random variable and let $X = f(U)$ and $Y = g(U)$ where $f, g : \mathbb{R} \to \mathbb{R}$ are non-decreasing functions. Show that $Cov(X, Y) \geq 0$.

6. Suppose that $X_1, \ldots, X_n$ is a random sample from a standard normal distribution and define the random variable

$$Y_n = \sum_{i=1}^{n} X_i^2.$$

(a) Show that $\sqrt{n/2}\,[\frac{1}{n}Y_n - 1]$ converges in distribution to the standard normal random variable Z.

(b) Let $U_n = (2n^2)^{-1/4}\sqrt{Y_n}$. Show that $\sqrt{n}\,[U_n - 2^{-1/4}]$ converges in distribution to a normal random variable with mean zero and variance $2^{-3/2}$.

7. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample of a normal distribution with mean zero and variance σ^2. Define

$$W_n = \frac{1}{\binom{n}{2}} \sum_{i=2}^{n} \sum_{j=1}^{i-1} |X_i - X_j|.$$

   (a) For $n = 4$, find b_1, b_2, b_3, b_4 such that $W_4 = b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + b_4 Y_4$, where $Y_1 \geq Y_2 \geq Y_3 \geq Y_4$ is the order statistics of X_1, X_2, X_3, X_4.

   (b) Show that $V_n = \frac{\sqrt{\pi}}{2} W_n$ is an unbiased estimator of σ.

Discrete Mathematics

1. (a) A line of n airline passengers is waiting to board a plane. They each hold a ticket to one of the n seats on that flight. (For convenience, let's say that the i th passenger in line has a ticket for the seat number i.) Unfortunately, the first person in line is crazy, and will ignore the seat number on their ticket, picking a random seat to occupy. All of the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random. Show that the number of sitting arrangements for which the last person sits on the first seat and the last person sits on the last seat are same (without explicitly counting the number).

   (b) Alice and Bob alternately choose numbers from among $1, 2, \cdots, 9$, without replacement. The first to obtain 3 numbers which sum to 15 wins. Does Alice (the first to play) have a winning strategy?

2. (a) N tigers and one sheep are put on a magic island that only has grass. Tigers can live on grass, but they want to eat sheep. If a tiger bites the sheep then it will become a sheep itself. If 2 tigers attack a sheep, only the first tiger to bite converts into a sheep. Tigers don't mind being a sheep, but they have a risk of getting eaten by another tiger. All tigers are intelligent and want to survive. Will the sheep survive?

   (b) Consider six distinct points in a plane. Let m and M denote respectively the minimum and the maximum distance between any pair of points. Show that $M/m \geq \sqrt{3}$.

3. (a) You are kept in a prison consisting of n^2 cells arranged like the squares of an $n \times n$ chessboard. There are doors between all adjoining cells. You are in one of the corner cells and it is told that you can get out of the prison provided you can get into the diagonally opposite corner cell after passing through *every other cell exactly once.* Can you obtain freedom?

(b) Inspector Bob has to put some criminals in cells of the prison. The criminals are notorious and can beat one another to death. If any criminal dies inside the cell, then the inspector will lose his job. In this scenario, the inspector thought of putting each criminal in a cell. But, his boss wants it to be done using the minimum number of cells. The only saving grace for the inspector is that the criminals fight according to the following pattern: (i) a criminal does not beat himself, (ii) if a criminal C_1 does not beat a criminal C_2, and criminal C_2 does not beat criminal C_3, then criminal C_1 does not beat criminal C_3 and vice-versa. Help the inspector by solving this problem efficiently.

4. (a) We define a set $\mathcal{S} \subset \mathbb{N}$, such that $|\mathcal{S}|$ is finite, to be "*crazy*" if $|\mathcal{S}| \in \mathcal{S}$. How many subsets of $\{1, 2, \ldots, n\}$ are there that are minimal crazy sets (minimality is defined in the sense that subsets that are crazy and do not properly contain any other crazy set). E.g., for a set $\{1, 2, 3\}$, the minimal crazy sets are $\{1\}$ and $\{2, 3\}$.

   (b) When n couples arrived at a party, they were greeted by the host and hostess at the door. After rounds of handshaking, the host asked the guests as well as his wife (the hostess) to indicate the number of hands each of them had shaken. He got $2n + 1$ different answers. Given that no one shook hands with his or her spouse, how many hands had the hostess shaken?

5. Tower of Hanoi (ToH) is a mathematical puzzle where we have three rods and n disks namely $d_1, \ldots, d_n$ (increasing order of size). The objective of the puzzle is to move the entire stack from rod 1 to rod 3, obeying the following simple rules:

   - Only one disk can be moved at a time.
   - Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack i.e. a disk can only be moved if it is the uppermost disk on a stack.
   - No disk may be placed on top of a smaller disk.

   Now consider the following problems and find the solution:

   (a) Assume that moving disk d_i from one rod to another takes i unit time. How much time is required to solve the problem?

   (b) Suppose you are not allowed to move any disk directly from rod 1 to rod 3, and vice versa (i.e. every move should involve rod 2). So, if you start with a single disk, it will take 2 moves (rod 1 to rod 2, then rod 2 to rod 3). Find the minimum number of moves required for $n = 2$, and then try to find an expression for the minimum number of moves required for general n.

6. (a) Show that there are exactly $(n + 1)$ way so that one can fill a bag with n fruits subject to the following constraints:

- The number of apples must be even.
- The number of bananas must be a multiple of 5.
- There can be at most 4 oranges.
- There can be at most 1 pear.

(b) Consider a building having a staircase with n stairs. In how many ways can a person climb the staircase, if she can climb by 1 or by 2 stairs in each step? Find out a closed form expressionin terms of n.

7. (a) On a table, a row of fifty coins of various denominations, is placed. A picks a coin from one of the ends and puts it in her pocket; then B chooses a coin from one of the ends and the alternation continues until B pockets the last coin. Devise a strategy for A so that she wins at least as much money as B.

   (b) Prove that if $n^2 + 1$ points are placed in an equilateral triangle (the region inside as well as the perimeter) of side length 1, then there are two points whose distance is at most $1/n$.

8. (a) A function $f : \{0,1\}^n \to \{0,1\}$ is called self-dual if $f(x_1, x_2, \ldots, x_n) = f(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$, for all $x_1, x_2, \ldots, x_n$, where $\overline{x_i}$ is complement of the bit x_i. Compute the number of self dual functions that are present over n tuples.

   (b) A 4×4 S-box is a permutation that takes 4-bit input and produces a 4-bit output. An S-box is called good if the S-box contains no fixed points (i.e. $\forall x \in \{0,1\}^4$, $S(x) \neq x$). Count the number of 4×4 good S-boxes.

Elementary Probability

1. (a) Given a biased coin with probability of head being $p \neq 1/2$, how can you generate a uniformly random bit string (i.e., you need to prove that the probability of each bit being 0 or 1 is 1/2).

   (b) Let us consider a seqeunce $\mathbf{Y} = (Y_1, \ldots, Y_q)$ of independent, and identically distributed uniform random variables over $\{0,1\}^n$. Let $\mathcal{L}$ be the following matrix of dimension $m \times q$ with rank r

$$\mathcal{L} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mq} \end{bmatrix}$$

   such that each $a_{ij} \in \{0,1\}^n$. For any vector $\mathbf{c} = (c_1, \ldots, c_m) \in (\{0,1\}^n)^m$, find the value of

$$\Pr[\mathcal{L} \cdot \mathbf{Y}^{\mathrm{T}} = \mathbf{c}^{\mathrm{T}}].$$

2. (a) Consider an one-dimensional X axis of length w units, i.e. the starting co-ordinate is $(0, 0)$ and the end co-ordinate is $(w, 0)$. A drunk guy is placed on the island at location $(n, 0)$ where $n < w$. He then randomly walks on the island along the X-axis: at each step, he either moves to the left or to the right from its current position each with probability 1/2. If he stands either on the location $(0, 0)$ or on the location $(w, 0)$, then he will immediately falls off. Can he surive?

   (b) Consider the above problem with X-axis is infinitely extended to the right direction and the drunk guy is placed at location $(1, 0)$. Then what is the probability that he will survive ?

   (c) Compute the expected lifespan of the drunk guy when he is placed at location $(1, 0)$ on the infinitely right extended X-axis.

3. (a) A line of 100 airline passengers is waiting to board a plane. They each hold a ticket to one of the 100 seats on that flight (for convenience, let's say that the nth passenger in line has a ticket for the seat number n). Unfortunately, the first person in line is crazy, and will ignore the seat number on his ticket, picking a random seat to occupy. All of the other passengers are quite normal, and will go to their proper seat unless it is already occupied. if it is occupied, they will then find a free seat to sit in, at random. What is the probability that the i^{th} person to board the plane will sit in his proper seat?

   (b) A machine randomly searches for integral solutions of the equation $x_1+x_2+x_3+x_4 = 17$, such that $|x_i| \leq 5$ for all $i \in \{1, 2, 3, 4\}$. What is the probability that it comes up with a solution with positive entries?

4. Refer to Figure 1. A secret agent, after finishing his job at A, wants to reach the extraction point at B using the grid lines. To reach B in minimal time, he can only move horizontally right or vertically up. His enemies are waiting at points P, Q and R. What is the probability that he will reach the extraction point without any obstacle?

5. (a) A man has ten coins. Nine ordinary and one with heads on both sides. He selects a coin at random, tosses it six times, and it always comes up heads. Compute the probability that he selected the double headed coin.

   (b) Two teams (Team A and Team B) play 5 match ODI series. Team A has a probability $\frac{3}{4}$ winning a single game. What is the probability Team A wins the series?

6. Suppose you and your friend have a two-sided coin each. Your coin lands Heads with probability $\frac{1}{6}$, while your friend's coin lands Heads with probability $\frac{3}{4}$. The two coins are independent of one another. Suppose you play a game where you both flip your coins once, and if they both land on the same side (i.e., both Heads or both Tails) you get x dollars from your

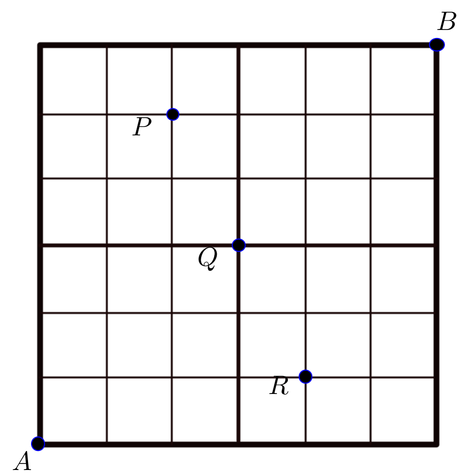


Figure 1

friend, but if they land on different sides, then your friend gets 2 dollars from you.

What is the minimum integer value of x for which your expected total winnings after 3 rounds of this game are positive (i.e., you are expected to make money rather than lose some)?

Basic Number Theory

1. (a) The general of a battelion A, let G_A, wants to inform the number of soldiers he has to the general of battelion B, let G_B. But G_A also wants to make sure that the enemy should not know the number of soldiers he has. For this, G_A came up with an idea. He asks his soldiers to line up in in rows of 11, then in rows of 17, 29, and 31. Respectively, each time, he noted down with remainder $8, 5, 16$, and 24. G_A passes the information $(8, 11), (5, 17), (16, 29)$ and $(24, 31)$. Can you say whether G_B can deduce the number of soldiers G_A has from the passed on information. If yes, then how many soldiers G_A has?

   (b) For every $m \in \mathbb{Z}^+$, we define a set $I_m = \{0, \ldots, m-1\}$. We define a function $\tau : I_a \times I_b \to I_{ab}$ as $(s, t) \mapsto (as + bt) \mod ab$. Prove that τ is bijective if and only if $\gcd(a, b) = 1$.

2. (a) If $n \geq 1$ is an integer, show that among $n,\ n+1,\ n+2$ and $n+3$, there is one which is co-prime to the other three.

   (b) Let $k > 1$ and $2^k - 1$ is a prime. Show that $n = 2^{k-1}(2^k - 1)$ is a perfect number. A perfect number is a positive integer that is equal to the sum of its positive divisors, excluding itself.

3. (a) Let p and q are large primes and $N = pq$. We choose a number e such that e is co-prime to $\phi(n)$ and $d = e^{-1} \mod \phi(n)$. Now we consider two functions $f_{N,e} : \mathbb{Z}_N \to \mathbb{Z}_N$, defined as $f_{N,e}(m) = m^e \mod N$ and $g_{N,d} : \mathbb{Z}_n \to \mathbb{Z}_N$ defined as $g_{N,e}(c) = c^d \mod N$. Prove that for $g_{N,e} \circ f_{N,e}$ is an identity function.

   (b) Let $\phi_2(n)$ is the number of positive integers a such that both a and $a + 1$ are co-prime to n. Find out the formula for $\phi_2(n)$. (You can assume $\phi_2(n)$ is a multiplicative function, i.e., if m and n are relatively prime then $\phi_2(mn) = \phi_2(m)\phi_2(n)$).

4. (a) There are n people numbered 1 to n standing in a circle. Starting the count with person number 1, every second person is eliminated until only one person is left. Where in the circle should a person stand to remain the last person standing?

   (b) Suppose there are 100 doors labelled with 1 to 100 and there are 100 persons labelled with 1 to 100. Initially each door is closed. For $1 \le i \le 100$, i-th person flips the state of the doors whose labels are multiple of i. Determine how many doors will remain open after 100-th person completes the task.

Calculus

1. Consider the function $f(x) = \begin{cases} px + q, & x > 0 \\ \cos 2x, & x \le 0 \end{cases}$

   (a) Find the values of the constants p and q for which the above function is differentiable, but not continuous.

   (b) Find the values of the constants p and q for which the above function is continuous, but not differentiable.

2. (a) Find the values of the constants a, b and c for which the following function is differentiable. (Give a and b in terms of c.)

$$f(x) = \begin{cases} cx^2 + 4x + 1, & x \ge 1 \\ ax + b, & x < 1 \end{cases}$$

   (b) Find all values of a, b such that $f'(x)$ is continuous:

$$f(x) = \begin{cases} ax + b, & x \ge 1 \\ x^2, & x < 1 \end{cases}$$

3. (a) Let a be a positive number. Then find out the value of

$$\lim_{n\to\infty} \left[\frac{1}{a+n} + \frac{1}{2a+n} + \ldots + \frac{1}{na+n} \right].$$

(b) Find out the value of

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^n-\left(1+\frac{1}{n}\right)\right\}^{-n}.$$

(c) Find out the value of

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\sqrt{4i/n.}\right],$$

where $[x]$ denotes the greatest integer which is less than or equal to x.

4. (a) Find out the value of the integral

$$\int_0^{\pi/4}\left[3\tan^2 x\right]dx.$$

(b) Prove that the integrals

$$\int_0^{\infty}\cos x^2 dx, \quad \text{and} \quad \int_0^{\infty}\sin x^2 dx$$

converges.

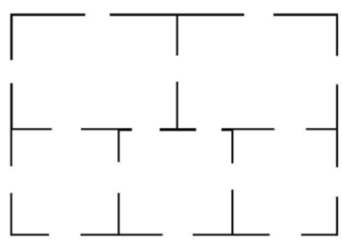
(c) Show that the derivative of an even function is odd and that the derivative of an odd function is even.

Group B

Graph Theory

1. Consider the following floor plan of five room research lab at IAI in Figure 1. Show that you can not find a continuous path that pass through each door exactly once. Now if you are allowed to close some doors of the lab, after closing at least how many doors you will find a continuous path that passes through each door exactly once?

Figure 2: Floor Plan of IAI Research Lab



2. (a) Construct a cubic graph with $2n$ vertices having no perfect matching. (A graph is cubic if every vertex has degree three.)

   (b) In a graph G with 10 vertices, among any three vertices of G, at least two are adjacent. Find the least number of edges that G can have. Can you draw such a graph?

3. (a) Prove that every simple graph $G = (V, E)$ has a bi-partite subgraph with at least $|E|/2$ edges.

   (b) Show that if a graph has $2n$ vertices and all of them have degree at least n, then the graph is connected.

4. (a) Two married couples want to cross a river. They can only use a boat that can carry one or two people from one shore to the other shore. Each husband is extremely jealous and is not willing to leave his wife with the other husband, either in the boat or on shore. How can these four people reach the opposite shore?

   (b) Let G be a simple graph with 19 edges, and degree of each vertex is greater than 3. Knowing nothing else about G, find (i) the maximum number of vertices that G could have, (ii) the maximum number of vertices that G could have for which one can conclude whether G is planar or not.

5. (a) Four people need to cross a rickety bridge at night. Unfortunately, they have only one torch and the bridge is too dangerous to cross

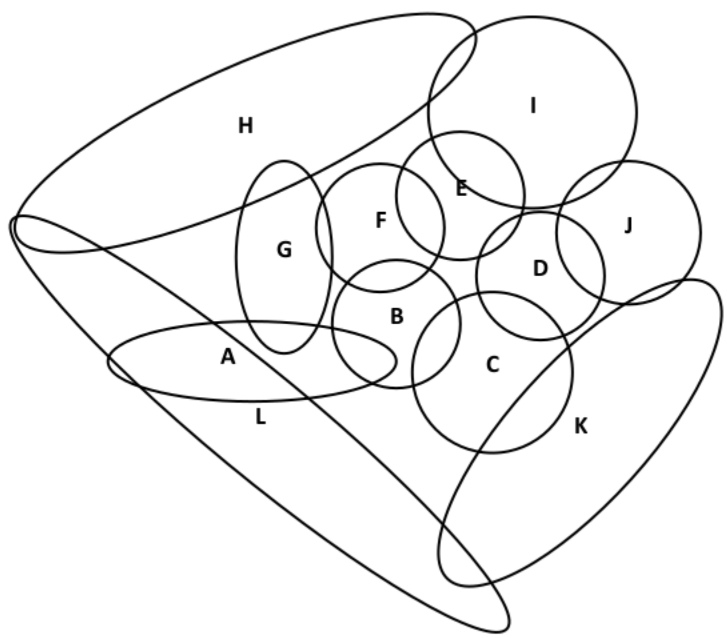
without one. The bridge is only strong enough to support two people at a time. Not all people take the same time to cross the bridge. Times for each person: 1 min, 2 mins, 7 mins and 10 mins. What is the shortest time needed for all four of them to cross the bridge?

(b) Can you tile a $m \times n$ checkerboard with dominoes (a domino being two adjacent squares)? Formulate this problem as a graph theoretic problem, and find the solution.

6. (a) Robot Sophia is walking on a *cyclic track*. The track is marked at evenly spaced intervals with 0s and 1s, with a total of 16 marks. Sophia can see the 4 marks closest to her. How should the 0s and 1s be put on the track so that she knows where on the track she is by just looking at the 4 closest marks?

   (b) Two player play a game on a graph G, alternatively choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceeding choices of the other player and hence they follow a path. The last person who is able to move, wins. Prove that the second player has a strategy to win if G has a perfect matching. Otherwise, first player wins.

7. There are 12 radio stations in India broadcasting music in local languages: A, B, C, D, E, F, G, H, I, J, K and L. Each radio station has a specific area in which people can listen to it, as drawn below. Unfortunately, some areas overlap. The officers of All India Radio can license specific wave fre-



quency to each station. But, they want to use the minimal amount of different frequencies, in order to minimize costs. Nevertheless, in an area of overlap, each radio station has to have different frequency; otherwise the listeners would not be able to listen to their station. Formulate the problem as a graph theoretic problem and find how to distribute the frequencies amongst the radio stations so that minimal amount of different frequencies is used?

Elements of Computing

1. (a) Given an array $A = \{a_1, a_2, \ldots, a_n\}$ of unsorted distinct integers, write a program in pseudo-code for the following problem: given an integer u, arrange the elements of the array A such that all the elements in A which are less than or equal to u are at the beginning of the array, and the elements which are greater than u are at the end of the array. You may use at most 5 extra variables apart from the array A.

   (b) How many asterisks ($*$) in terms of k will be printed by the following C function, when called as $count(m)$, where $m = 3^k$? Assume that 4 bytes are used to store an integer in C and k is such that 3^k can be stored in 4 bytes.

```
void count(int n)
{
   printf("*");
   if(n>1)
   {
       count(n/3);
       count(n/3);
       count(n/3);
   }
}
```

2. (a) Consider the pseudo-code given below.

   Input: integers b and c.

```
1. a_0 = max(b,c),a_1 = min(b,c)
2. i = 1.
3. Divide a_{i-1} by a_i.
4. Let q_i be the quotient and r_i be the remainder.
5. If r_i==0 then go to Step 9.
6. a_{i+1} = a_{i-1}-q_i*a_i.
7. i = i + 1
8. Go to Step 3.
9. Print a_i.
```

What is the mathematical relation between the output `a_i` and the two inputs `b` and `c`.

(b) You are given the following file abc.h:

```
#include <stdio.h>
#define SQR(x) (x*x)
#define ADD1(x) (x=x+1)
#define BeginProgram int main(int argc,char *argv[]){
#define EndProgram return 1;}
```

For each of the following code fragments, what will be the output?

```
(i)  #include "abc.h"
main()
{
    int y = 4;
    printf("%d\n", SQR(y+1));
}
```

```
(ii)  #include "abc.h"
BeginProgram
int y=3;
printf("%d\n", SQR(ADD1(y)));
EndProgram
```

3. (a) Consider the following program:

```
void g(char *s, int len)
{
    if(len == 0)
         return;
    char temp = s[0]; s[0] = s[len-1]; s[len-1] = temp;
    g(s+1, len-2);
}


int main()
{
    char s[] = "hello";
    g(s, strlen(s));
    printf("%s", s);
}
```

What will be the output of the above program ?

(b) Let A be an integer array of size N. The function max returns the maximum of its two arguments. The two code fragments shown below are equivalent (i.e., n, r, s have the same value at the end) if we fill the blank with (i).

```
n = 0, r = 0, s = 0;
while (n != N)
{
    s =  max(s + A[n], 0);
    r = max(r, s);
    n = n+1;
}
```

```
n = -1, r = 0, s = 0;
while (n != N)
{
    r = max(r, s);
    n = n+1;
    if(____(i)____)
    s =  max(s + A[n], 0);
}
```

(c) When called with $x = 10! + 9!$ and $y = 8!$, what does the function f below return?

```
int f(x,y)
{
    int count=0;
    while(x > y)
    {
        x=x-y;
        count++;
    }
    while(y > x)
    {
        y=y-x;
        count++;
    }
    return count;
}
```

Data Structures

1. You are given k sorted lists, each containing m integers in ascending order. Assume that (i) the lists are stored as singly-linked lists with one integer in each node, and (ii) the head pointers of these lists are stored in an array.

   (a) Write an efficient algorithm that merges these k sorted lists into a single sorted list using $\Theta(k)$ additional storage.

(b) Next, write an efficient algorithm that merges these k sorted lists into a single sorted list using $\Theta(1)$ additional storage.

(c) Analyse the time complexity of your algorithm for both the cases.

2. (a) Given a data structure D supports the following sequence operations:

$$\mathsf{D.insert\_first(x), D.delete\_first(), D.insert\_last(x), D.delete\_last()},$$

each in $O(1)$ time. In addition, D also supports the operations

$$\mathsf{D.insert\_at(x,i), D.delete\_at(i)},$$

both of which requires $O(\log n)$ time. Device efficient algorithms to implement the following higher level operations using the above lower-level operations:

i. $\mathsf{reverse}(D, i, k)$: Reverse in D the order of the k items starting at index i.

ii. $\mathsf{shift\_left}(D, k)$: Move the first k items in order to the end of the sequene in D.

iii. $\mathsf{move}(D, i, k, j)$: Move the k items in D starting at index i, in order, to be in front of the item at index j. Assume that the inequalit $i \leq j < i + k$ does not hold.

Compute the time complexity of each proposed algorithms. Assume that all the delete operations return the value deleted.

(b) An integer array A is called (n, k)-type if (i) A contains n distinct integers, (ii) exactly k of them are even, and (iii) the odd integers in A appear in sorted order. Design an efficient algorithm to sort the elements of an $(n, \lceil n/\lg n \rceil)$-type array. Compute the time complexity of your algorithm.

3. (a) Consider a linked list containing n nodes, where each node contains two pointers ptr1 and ptr2. For each node, ptr1 points to the next node of the list. Describe how pointer ptr2 should be set up for each node so that you will be able to locate the i-th node from the start node in the list traversing no more than $\lceil \log i \rceil + \lceil i/2 \rceil$ nodes.

(b) Give an efficient implementation for a data structure STACK_MIN to support an operation Min that reports the current minimum among all elements in the stack. Usual stack operations (Create, Push, Pop) are also to be supported.

4. (a) Let H_1 and H_2 be two complete binary trees that are heaps as well. Assume H_1 and H_2 are max-heaps, each of size n. Design and analyze an efficient algorithm to merge H_1 and H_2 to a new max-heap H of size $2n$.

(b) Let B be a rooted binary tree of n nodes. Two nodes of B are said to be a sibling pair if they are the children of the same parent. Design an $O(n)$ time algorithm that prints all the sibling pairs of B.

5. Given an airport with a single runway, design an efficient runway reservation system of that airport. Each reservation request comes with requested landing time let's say t. Landing can go through if there is no landing scheduled within k minutes of requested time, that means t can be added to the set of scheduling landings. The parameter k can vary and depends on external conditions. This system helps with reservations for the future landings. Once the plane lands safely, you have to remove the plane for landing sets.

6. Let T be an AVL tree for storing a set of n integers. Insertions and deletions in T can hence be done in $O(\log n)$ time. Given two integers a and $b, a < b$, you have to output n_{ab}, the number of integers in T whose value lies within $[a; b]$ in $O(\log n)$ time. For this purpose, what modification of T and its insertion algorithm are required? Give a pseudocode for computing n_{ab}.

Design and Analysis of Algorithms

1. (a) Give a strategy to sort four distinct integers a, b, c, d in increasing order that minimizes the number of pairwise comparisons needed to sort any permutation of a, b, c, d.

   (b) The vertices of a triangle T are given. For an arbitrary point P in the plane, give an algorithm to test if P belongs to the interior of T. (The interior of T does not include its edges).

2. (a) You are given an unordered sequence of n integers with many duplications, such that the number of distinct integers in the sequence is $O(\log n)$. Design a sorting algorithm and its necessary data structure(s), which can sort the sequence using at most $O(n \log(\log n))$ time. Justify the time complexity of your proposed algorithm.

   (b) You are given two strings S and T, each of length n, consisting only of lower case English letters $(a, b, \ldots, z)$. Propose an $O(n)$-time algorithm to decide whether S can be obtained by permuting the symbols of T.

3. Let M be an $(n \times n)$ matrix where each element is a distinct positive integer. Construct another matrix M' by permuting the rows and/or permuting the columns, such that the elements ofone row appear in increasing order (while looking from left to right) and those of one column appear in decreasing order (while looking from top to bottom). Describe an $O(n^2)$ time algorithm for constructing M'. Justify your analysis.

4. A connected, simple, undirected planar graph $G(V, E)$ is given where V denotes the set of vertices and E denotes the set of edges. In V, there is a designated source vertex s and a designated destination vertex t. Let

$P(v)$ denote the shortest path (may contain repetition of nodes/edges) from s to t that passes through v, and let $l(v)$ denote the path length (i.e., the number of edges)of $P(v)$. Describe an $O(|V|)$ time algorithm that determines the value of τ, where $\tau = \max_{\forall v \in V} l(v)$. Justify your analysis.

5. Devise an algorithm to color the edges of any bipartite graph of maximum degree Δ with exactly Δ many distinct colors.

Circuits and Systems

1. (a) Consider the multiplication of two 2-bit integers a_1a_0 and b_1b_0 to get a 4-bit output $c_3c_2c_1c_0$. Assuming that the right most bit is the least significant bit, derive Boolean functions for c_0 and c_3.

   (b) Design a combinational logic circuit that takes an unsigned 2-bit integer as input and computes its square.

2. (a) You are given a logic block L that takes two inputs A and B, and produces $\overline{A}+B$ as output. Realize a two-input XOR gate using only the logic block L. You can use as many pieces of block L as you need, and the constant function 0; but no other type of gate.

   (b) A binary sting $a_na_{n-1}\ldots a_0$ is called palindrome if $a_na_{n-1}\ldots a_0 = a_0a_1\ldots a_n$. Design a combinatorial circuit over 4-bit inputs that outputs 1 if and only if the input is a palindrome.

3. (a) You are required to design a 4-bit prime number checker. Note that 0 and 1 are not prime. Design the circuit using a single 4×1 multiplexer and a minimal number of AND, OR or NOT gates.

   (b) Consider you have a number of comparator circuits. Design a digital circuit that takes n inputs, each of which is a 4 bit number and output the minimum number. You may use basic combinatorial gates in addition to the comparator circuit.

   (c) A Boolean function g is said to be the dual of another Boolean function f if g is obtained from f by interchanging the operations $+$ and ., and the constants 0 and 1. A Boolean function f is self-dual if $f = g$. Given $f(a,b,c) = a\overline{b} + \overline{b}c + x$, find the Boolean expression x such that f is self-dual.

4. (a) Let $a_{n-1}a_{n-2}\ldots a_0$ and $b_{n-1}b_{n-2}\ldots b_0$ denote the two's complement representation of two integers A and B respectively. Addition of A and B yields a sum $S = s_{n-1}s_{n-2}\ldots s_0$. The outgoing carry generated at the most significant bit position, if any, is ignored. Show that an overflow (incorrect addition result) will occur only if the following Boolean condition holds: $\overline{s}_{n-1} \oplus (a_{n-1}s_{n-1}) = b_{n-1}(s_{n-1} \oplus a_{n-1})$.

(b) Design a digital circuit to compare two three bit numbers $A(A_2A_1A_0)$ and $B(B_2B_1B_0)$; the circuit should have three output terminals indicating $A = B,\ A < B$ and $A > B$. Draw the corresponding circuitry.

5. A choke coil connected across a 500 V, 50 Hz supply takes 1 A current at a power factor of 0.8.

   (a) Determine the capacitance that must be placed in series with the choke coil so that it resonates at 50 Hz.

   (b) An additional capacitor is now connected in parallel with the above combination in (a) to change the resonant frequency. Obtain an expression for the additional capacitance in terms of the new resonant frequency.

6. (a) An arc lamp requires a direct current of 10 A at 80 V to function. Prove that if it is connected to a 220 V (rms), 50 Hz AC supply, the series inductor needed for it to work is close to 0.065 H.

   (b) A parallel plate capacitor is charged to 75 μC at 100 V. After removing the 100 V source, the capacitor is immediately connected to an uncharged capacitor with capacitance twice that of the first one. Determine the energy of the system before and after the connection is made. Assume that all capacitors are ideal.

Group C

Topology

1. Let X and Y be non-empty subsets of a metric space M. Define $d(X,Y) = \inf\{d(x,y)|x \in X, y \in Y\}$.

   (a) Suppose X contains one point x and Y is closed. Prove that $d(X,Y) = d(x,y)$ for some $y \in Y$.

   (b) Suppose X is compact and Y is closed. Prove $d(X,Y) = d(x,y)$ for some $x \in X$ and $y \in Y$.

2. Let $A \subseteq \mathbb{R}$ be uncountable. Recall that $p \in A$ is an accumulation point if $(B_\epsilon(p) \setminus p) \cap A \neq \phi \forall \epsilon > 0$.

   (a) Show that A has at least one accumulation point.

   (b) Show that A has uncountably many accumulation points.

3. Show that the set $\overline{\{(x, \sin(1/x))|x \in (0, 2\pi]\}}$ is connected but not path connected.

4. (a) Let $f : [a,b] \to \mathbb{R}$ be a continuous real function (in the usual sense) on a closed bounded interval. Prove that its graph: $\Gamma = \{(x, f(x)) : a \leq x \leq b\}$ is closed in $\mathbb{R}^2$ with the usual topology.

   (b) For the real function $f : \mathbb{R} \to \mathbb{R}$ given by: $f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ determine the closure of its graph.

5. Given subsets A, B of a topological space, prove the following for their closures A, B:

   (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   (b) $\overline{A \times B} = \overline{A} \times \overline{B}$.

6. (a) Let $\mathbb{D}^2 = \{x \in \mathbb{R}^2, |x| \leq 1\}$ and $\mathbb{S}^1 = \{x \in \mathbb{R}^2, |x| = 1\}$. Show that $X = \mathbb{R}^2 \setminus \mathbb{D}^2$ is homeomorphic to $\mathbb{S}^1 \times (0,1)$.

   (b) Identify the x-axis with $(-\infty, +\infty)$. Is $X \setminus ([1, +\infty) \cup [-1, -\infty))$ connected? Is X homeomorphic to $\mathbb{R}^2$?

7. Recall that $GL_2(\mathbb{R})$ denotes the set of all invertible 2×2 matrices and $O(2)$ denotes the set of all 2×2 orthogonal matrices. Consider the usual topology on them obtained as a subset of $\mathbb{R}^4$.

   (a) Is $GL_2(\mathbb{R})$ compact? Is it path connected?

(b) Show that $O(2)$ is homeomorphic to $\mathbb{S}^1 \times \{0,1\}$.

8. (a) Let $X = \mathbb{R}^3 \setminus \mathbb{S}^1$ and $Y = \mathbb{R}^3 \setminus \mathbb{R}$. Are X and Y homeomorphic? Are they homotopic?

   (b) Define $Z = \mathbb{S}^1 \times [0,1]$. Let $\tilde{Z}$ be the quotient space obtained from Z by identifying the points $\{\theta\} \times \{0\}$ and $\{\theta\} \times \{1\}$ for some fixed $\theta \in \mathbb{S}^1$. Find $\pi_1(\tilde{Z})$.

9. (a) Recall that given $X \subset Y$, a retraction $r : Y \to X$ is a continuous map such that $r|_X = id_X$. Let $A = \mathbb{S}^1 \times \mathbb{D}^2$ and ∂A be the boundary of A (homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$). Prove that there is no retraction from A to ∂A.

   (b) Let $X = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1, 0 \le z \le 1\}$ and $Y = \{(x,y,0) \in \mathbb{R}^3 : 1 \le x^2+y^2 \le 4\}$. Write an explicit homeomorphism between X and Y.

10. (a) Let $X = \mathbb{R}^3 \setminus \mathbb{S}^1$ and $Y = \mathbb{R}^3 \setminus \mathbb{R}$. Are X and Y homeomorphic? Are they homotopic?

    (b) Let $X = \mathbb{R}^3 - E$, where E is the union of the x-axis and the y-axis. Let $x_0 \in X$ be any point. Compute the fundamental group $\pi_1(X, x_0)$.

11. (a) Define $Z = \mathbb{S}^1 \times [0,1]$. Let $\tilde{Z}$ be the quotient space obtained from Z by identifying the points $\{\theta\} \times \{0\}$ and $\{\theta\} \times \{1\}$ for some fixed $\theta \in \mathbb{S}^1$. Find $\pi_1(\tilde{Z})$.

    (b) Recall that given $X \subset Y$, a retraction $r : Y \to X$ is a continuous map such that $r|_X = id_X$. Let $A = \mathbb{S}^1 \times \mathbb{D}^2$ and ∂A be the boundary of A (homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$). Prove that there is no retraction from A to ∂A.

12. (a) Let $X = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1, 0 \le z \le 1\}$ and $Y = \{(x,y,0) \in \mathbb{R}^3 : 1 \le x^2+y^2 \le 4\}$. Write an explicit homeomorphism between X and Y.

    (b) Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. Let $p = (0,0,1)$, $q = (0,0,-1) \in S^2$. Prove that $S^2 - \{p,q\}$ is not contractible.

Analysis

1. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = (0,0)$.

   (a) Find $\partial_x f(x,y)$ and $\partial_y f(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

   (b) Is f continuous at $(0,0)$?

2. (a) Investigate convergence/divergence of $\sum_n \frac{1}{1+z^n}$ for complex values of z.
   (b) Consider a sequence $\{s_n\}$ of non-negative real numbers. Show that divergence of $\sum \frac{\sqrt{s_n}}{n}$ implies divergence of $\sum s_n$.

3. Let h be a continuous function on $[0,1]$
   (a) Evaluate $\lim\limits_{n\to\infty} \int_0^1 x^n h(x)dx$.
   (b) Evaluate $\lim\limits_{n\to\infty} n \int_0^1 x^n h(x)dx$.

4. (a) For $x \in [0, \frac{\pi}{2}]$ show that $sinx \geq \frac{2}{\pi}x$.
   (b) Show that if $\lambda < 1$, then $\lim\limits_{L\to\infty} L^\lambda \int_0^{\frac{\pi}{2}} e^{-L\sin t}dt = 0$.

5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function with period 1. Prove the followings.
   (a) ϕ is a bounded function with maxima and minima.
   (b) ϕ is uniformly continuous on $\mathbb{R}$.
   (c) There exists $x_0 \in \mathbb{R}$ such that $\phi(x_0 + \pi) = \phi(x_0)$.

6. Consider the sequence $\{f_n\}$ defined by $f_n(x) = \frac{1}{1+x^n}$ for $x \in [0,1]$.
   (a) Find $f(x) = \lim\limits_{n\to\infty} f_n(x)$.
   (b) Show that for $0 < a < 1$, $\{f_n\}$ converges uniformly to f on $[0,a]$.
   (c) Show that $\{f_n\}$ does not converge uniformly to f on $[0,1]$.

7. (a) Let $\gamma : [0,1] \to \mathbb{C}$ be given as $\gamma(t) = 1 + \mathrm{i}t$. Calculate

$$\int_\gamma \frac{1}{z}\text{log}z dz$$

   .
   (b) Let f and g be entire functions such that $|f(z)| < |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists a constant $\lambda \in \mathbb{C}$ such that $f(z) = \lambda g(z)$ for all $z \in \mathbb{C}$.

8. (a) Find maximum of $|f(z)|$ on $|z| \leq 1$, when $f(z) = z^2 - 3z + 2$.
   (b) Let $C = \{z \in \mathbb{C} : |z| = 5\}$. What is value of M such that $2\pi\mathrm{i}M = \int_C \frac{1}{z^2-5z+6}dz$?

9. (a) Let $p(z)$ be a polynomial in the complex variable z with real coefficients. Prove that roots of $p(z) = 0$ appear in conjugate pairs.
   (b) Let $T(z) = \frac{z}{z+1}$. Find the inverse image of the disk

$$D = \{z \in \mathbb{C} : |z| < 1/2\}$$

   under T.

10. (a) Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \begin{cases} \frac{\bar{z}^3}{z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Prove that the complex derivative $f'(z)$ does not exist for any $z \in \mathbb{C}$.

(b) Find the contour integral $\int_\gamma \bar{z} dz$ where γ is the triangle ABC oriented clockwise, with vertices $A = 0$, $B = 1 + i$ and $C = -2$.

11. (a) Find a curve C in $\mathbb{R}^2$, passing through the point $(3, 2)$, with the following property: Let $L(x_0, y_0)$ be the segment of the tangent line to C at (x_0, y_0) which lies in the first quadrant. Then each point (x_0, y_0) of C is the midpoint of $L(x_0, y_0)$.

(b) Discuss the solvability of the differential equation

$$(e^x sin\ y)((y')^3) + (e^x cos\ y)y' + e^y tan\ x = 0$$

with the initial condition $y(0) = 0$.

Linear Algebra

1. (a) Given a nonzero column vector $A = [a_1\ a_2\ \cdots\ a_n]^T$. Find the non-zero eigenvalues and eigenvectors for AA^T.

(b) Let A and B be two $n \times n$ matrices. Show that AB and BA have the same eigen values.

2. (a) Let A, B, C be $n \times n$ real square matrices. If $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is diagonalizable, show that A and C are also diagonalizable.

(b) Find all $n \times n$ matrices A of real numbers that satisfy $A^T = ABA^{-1} - B$ for some $n \times n$ matrix B of real numbers.

3. Let $U(n)$ be the set of $n \times n$ complex matrices A satisfying $AA^* = Id$ and $SU(n) = \{A \in U(n) | det(A) = 1\}$. Determine the centers of $U(n)$ and $SU(n)$ for each $n \in \mathbb{N}$.

4. Let V be the linear space of all polynomials $p(x)$ of degree $\leq n$. Let $T(p(x)) := p''(x)$.

(a) Show that T is a linear transformation.

(b) Determine the rank and nullity of T.

5. (a) Let $\mathcal{S}$ be the vector space of all real–valued sequences over $\mathbb{R}$. Let $\{e_n^{(i)}\}_n$ be the sequence such that $e_i^{(i)} = 1$ and $e_n^{(i)} = 0$ for $n \neq i$. Is the set $\{\{e_n^{(i)}\}_n : i \in \mathbb{N}\}$ a basis of $\mathcal{S}$?

(b) Let A be $n \times n$ matrix of complex numbers and $A^2 = A^* A$. Show that A is a Hermitian matrix.

Abstract Algebra

1. For each question, compute the orders of the given groups say if those two given groups are isomorphic or not

   (a) $\mathbb{Z}/84\mathbb{Z} \times \mathbb{Z}/360\mathbb{Z}$ and $\mathbb{Z}/840\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$.

   (b) $\mathbb{Z}/495\mathbb{Z} \times \mathbb{Z}/105\mathbb{Z}$ and $\mathbb{Z}/675\mathbb{Z} \times \mathbb{Z}/77\mathbb{Z}$.

2. (a) Let a, b, c be three elements of the finite field $\mathbb{Z}_p$ of p elements, where p is a prime of the form $3k+2$. Prove that, $a^2+b^2+c^2-ab-bc-ca = 0$ if and only if $a = b = c$.

   (b) Prove that the polynomial x^7+ax+b does not split into linear factors (not necessarily distinct) over $\mathbb{Z}_{23}$ (the field with 23 elements) for any $(a, b) \neq (0, 0)$.

3. (a) Let $f(x, y, z, w)$ be a real polynomial (polynomial with real coefficients) in four variables x, y, z and z. Construct a real polynomial $F(x, y, z, w)$ such that, F is divisible by f in the polynomial ring $\mathbb{R}[x, y, z, w]$ and any monomial in F has degree divisible by 5.

   (b) A positive integer n is said to be good, if whenever n divides $a + b + c + ab + bc + ca + abc$ for some positive integers a, b and c, then n must divide at least one of them. Determine all good integers.

4. (a) Let A and B be two groups. Let C be a subgroup of A and D be a subgroup of B. Show that $(A \times B)/(C \times D) \cong A/C \times B/D$.

   (b) Prove that the set of all inner automorphisms of a group G is a normal subgroup subgroup of $Aut(G)$.

5. (a) Prove that every PID is Noetherian. Give an example of a non Noetherian UFD.

   (b) Suppose $F \subset K$ is a field extension and $\alpha \in K$. If $[F[\alpha] : F] = 5$, prove that $F[\alpha^2] = F[\alpha]$.

   (c) Which ideals in the polynomial ring $R := \mathbb{C}[x, y]$ contain $f_1 = x^2 + y^2 - 5$ and $f_2 = xy - 2$?

6. (a) Find all group homomorphisms from $(\mathbb{Z}_8, +) \to (\mathbb{Z}_6, +)$.

   (b) Show that every group of order 15 is cyclic.

7. (a) Show that $x^2 + x + 4$ is irreducible over $\mathbb{Z}_{11}$.

   (b) Show that $\mathbb{Z}_{11}[x]/ < x^2 + x + 4 >$ is a field and find its number of elements.