Quantum heat exchange fluctuation relation

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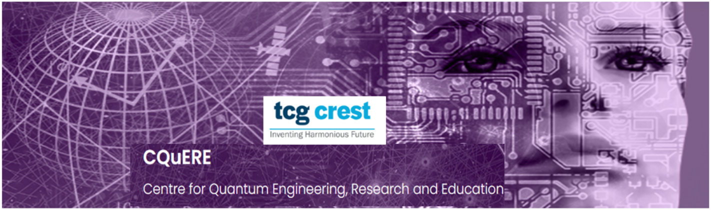


International Symposium on Quantum Information and Communication (ISQIC), 2025

CQuERE, TCG CREST

Kolkata, India

31st March – 2nd April, 2025

Sketch

- A bird's eye view of Thermodynamics
- Fluctuation Theorems
- Jarzynski-Wójcik heat-exchange relation
- Wigner function approach
- Discussion

Thermodynamics

The role of quantum information in thermodynamics – a topical review, J. Goold et. al, J. Phys. A, **49** 143001 (2016)

- If physical theories were *people*, THERMODYNAMICS would be the *village witch*!
- Einstein: "... the only theory with universal content, which I am convinced that within the framework of applicability of its basic concepts will never be overthrown"

Thermodynamics

Towards the first law

Leonardo da Vinci (1452-1519):
The French academy must refuse
all proposals for perpetual motion

Prohibitio ante legem

Steam age

Industrial revolution:
England becomes world power

1769 James Watt: patent on steam engine
improving design of Thomas Newcomen

Sidi Carnot (1796-1832)
Military engineer in army Napoleon

1824: Reflexions sur la Puissance Motrice du Feu, et sur les Machines Propres a Developer cette Puissance:

"The superiority of England over France is due to its skills to use the power of heat"

The First Law

Julius Robert von Mayer	(1814-1878)	1842
James Prescott Joule	(1818-1889)	1847
Herman von Helmholtz	(1821-1894)	1847
Germain Henri Hess	(1802-1850)	1840

Heat and work are forms of energy

Energy is conserved: $dU = dQ + dW$
heat + work added to system

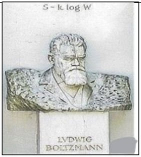
The Second Law

Rudolf Clausius (1822-1888)
1865: **Entropy** related to Heat:
Clausius inequality: $dS \geq dQ/T$

Clausius formulation:
Heat goes from high to low temperature

Most common formulation:
Entropy of a closed system cannot decrease

Ludwig Boltzmann (1844 -1906)
Statistical thermodynamics
S = k Log W



Gibbs

Josiah Willard Gibbs (1839-1903)
Papers in 1875,1878

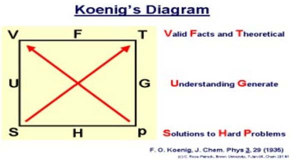
Thermodynamics according to Clausius:

Die Energie der Welt ist konstant;
die Entropie der Welt strebt einen Maximum zu.

The energy of the universe is constant;
The entropy of the universe approaches a maximum.

Thermodynamics

- Equilibrium thermodynamics formulates universally valid statements based on phenomenological observation.
- Reforumulation of such statements is required when the system is driven off from equilibrium.
- Stochastic fluctuations begin to impact the laws of thermodynamics when we move away from equilibrium.



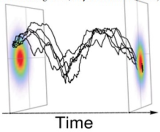
Statical Thermodynamics

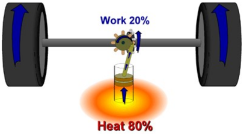
- Thermodynamic quantities of interest: work W, heat Q and entropy S.
- Classical statistical thermodynamics: W, Q, S along phase-space trajectories can be defined.
- Quantum thermodynamics: Phase-space trajectories?
- Two-time measurements (initial and final) are employed.

Quantum Thermodynamics

- New precision experimental techniques allow exploration of the quantum foundations of thermodynamics
- Testing the limits set by theory, experimental designs to build tiny engines (powered by a few quantum systems) & measuring any feeble signal from it is possible.

T. M. Hoang *et al.*, Phys. Rev. Lett. (2018)

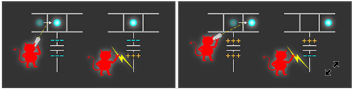




Quantum Thermodynamics

On-Chip Maxwell's Demon as an Information-Powered Refrigerator, J. V. Koski *et. al*, Phys. Rev. Lett. **115**, 260602 (2015)

- **Maxwell's demon faces the heat!! Jonne Koski and colleagues designed the laboratory equivalent of the Maxwell Demon (at Aalto University in Finland). The operation of the demon is directly observed as a temperature drop in the system; a simultaneous temperature rise in the demon too is recorded confirming the thermodynamic cost.**
- **This test of second law of thermodynamics confirmed that manipulating energy comes with a price!**



APS/Alan Stonebraker

Figure 1: An electronic version of a self-contained (autonomous) Maxwell demon. The "system" is a single-electron box connected to an external potential. The demon monitors the charge on the box. (Left) If an electron (blue) enters the box, the demon immediately traps it by applying a positive charge. (Right) If the electron leaves the box, the demon repels it by applying a negative charge. This is the electronic equivalent of the demon opening or shutting the door for fast/slow particles in Maxwell's original thought experiment.

Definitions

- **For a quantum system in a state ρ and with a Hamiltonian H, Internal Energy (i.e, average energy) is defined by $\langle E\rangle = U(\rho) = \mathrm{Tr}[\rho\, H]$.**
- **Average Heat:** $\langle Q\rangle = \int_0^\tau \mathrm{Tr}[\dot{\rho}(t)\, H(t)]\, dt$
- **Average work done:** $\langle W\rangle = \int_0^\tau \mathrm{Tr}[\rho(t)\, \dot{H}(t)]\, dt$
- **Change in internal energy: (First law)**

$$\begin{aligned}\Delta U &= \mathrm{Tr}[\rho(\tau)\, H(\tau)] - \mathrm{Tr}[\rho(0) H(0)] \\ &= \int_0^\tau \frac{d}{dt}\left(\mathrm{Tr}[\rho(t)\, H(t)]\right)\, dt \\ &= \langle Q\rangle + \langle W\rangle.\end{aligned}$$

- **Entropy:** $S = \mathrm{Tr}[\rho\, \ln\, \rho]$
- **Free Energy– relative to a thermal bath at temperature T :**

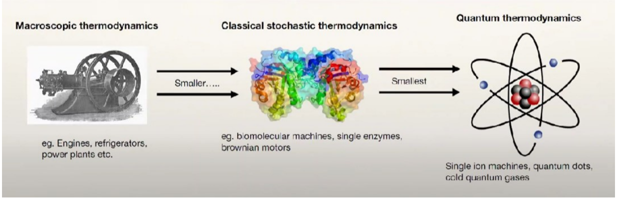
$$F = U - T\, S$$

What are Fluctuation Relations?

- They describe non-equilibrium transformation of a thermodynamic system.
- They constitute refinement of second law of thermodynamics.
- They connect the probabilities for quantities like work, heat, entropy to their counterparts in the time-reversed set-up.
- They are derived based on the mathematical framework describing the thermodynamic properties of microscopic systems.

This talk: Heat-exchange fluctuation relation

- Jarzynski-Wójcik heat-exchange fluctuation relation based on Wigner function approach.



Overview

- Exchange-fluctuation theorems (XFT) involving thermodynamic quantities like work, heat, entropy have been proposed during the last two decades.

Milestone

Nonequilibrium Equality for Free Energy Differences

C. Jarzynski
Phys. Rev. Lett. **78**, 2690 – Published 7 April 1997

$$\left\langle e^{-\beta W} \right\rangle = e^{-\beta \Delta F}$$

C. Jarzynski, PRL **78**, 2690 (1997)

50 years PRL moving physics forward

Classical stochastic dynamics from time 0 to τ
in contact with a heat bath at temperature $\beta=(k_B T)^{-1}$

PHYSICAL REVIEW E VOLUME 60, NUMBER 3 SEPTEMBER 1999

Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences

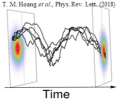
Gavin E. Crooks*
Department of Chemistry, University of California at Berkeley, Berkeley, California 94720
(Received 17 February 1999)

There are only a very few known relations in statistical dynamics that are valid for systems driven arbitrarily far-from-equilibrium. One of these is the fluctuation theorem, which places conditions on the entropy production probability distribution of nonequilibrium systems. Another recently discovered far from equilibrium expression relates nonequilibrium measurements of the work done on a system to equilibrium free energy differences. In this paper, we derive a generalized version of the fluctuation theorem for stochastic, microscopically reversible dynamics. Invoking this generalized theorem provides a succinct proof of the nonequilibrium work relation. [S1063-651X(99)10109-0]

PACS number(s): 05.70.Ln, 47.52.+j, 82.20.Mj

$$\frac{P_F(+\omega)}{P_R(-\omega)} = e^{+\omega}.$$

T. M. Hoang *et al.*, Phys. Rev. Lett. (2018)



Jarzynski-Wójcik XFT

VOLUME 92, NUMBER 23 PHYSICAL REVIEW LETTERS week ending 11 JUNE 2004

Classical and Quantum Fluctuation Theorems for Heat Exchange

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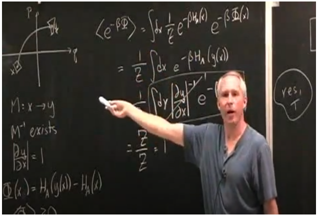
(Received 22 October 2003; published 11 June 2004)

The statistics of heat exchange between two classical or quantum finite systems initially prepared at different temperatures are shown to obey a fluctuation theorem.

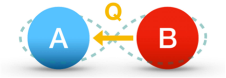


Jarzynski-Wójcik Heat Exchange Fluctuation Theorem

- Jarzynski-Wójcik XFT:

$$\ln\left[\frac{p_\tau(+\mathcal{Q})}{p_\tau(-\mathcal{Q})}\right] = \triangle\beta\,\mathcal{Q}, \quad \triangle\beta = \frac{1}{k\,T_A} - \frac{1}{k\,T_B}$$

k : Boltzmann constant; $\mathcal{Q}$: Amount of heat exchanged.



Jarzynski-Wójcik XFT

- Jarzynski-Wójcik XFT:

$$\ln\left[\frac{p_\tau(+\mathcal{Q})}{p_\tau(-\mathcal{Q})}\right] \;=\; \triangle\beta\,\mathcal{Q}, \quad \triangle\beta = \frac{1}{k\,T_A} - \frac{1}{k\,T_B}$$

- Quantifies the ratio of probability $p_\tau(+\mathcal{Q})$ of heat exchange during interaction of systems A and B for a fixed time duration τ, to its time-reversed counterpart $p_\tau(-\mathcal{Q})$.

Thermodynamic arrow of heat flow

- Clausius' Inequality:

$$Q_A\left(\frac{1}{T_A}-\frac{1}{T_B}\right)\geq 0$$

- Q_A=Heat flow into system A.
- If $T_A < T_B \Rightarrow \; Q_A > 0$.
- Heat is transferred from hot (B) to cold (A) system.

Jarzynski-Wójcik fluctuation theorem

- The Jarzynski-Wójcik XFT

$$\frac{p_\tau(+\mathcal{Q})}{p_\tau(-\mathcal{Q})} = e^{\triangle\beta\,\mathcal{Q}}, \quad \triangle\beta = \frac{1}{k\,T_A} - \frac{1}{k\,T_B}$$

  quantifies the *relative likelihood* of heat exchange process and its time-reversed twin, and it shows that *heat flow from a colder to a hotter object is exponentially suppressed.*
- Strict directionality of thermodynamic heat flow forms the foundational features of this XFT relation.

Classical phase-space description for Jarzynski-Wójcik XFT relation

- Jarzynski and Wójcik considered two systems A and B, phase-space evolution of which is governed by Hamiltonians $H_A(\xi_A)$ and $H_B(\xi_B)$
- ξ_A, ξ_B denote phase-space variables (e.g., positions and momenta) of systems A and B.

Classical phase-space description for Jarzynski-Wójcik XFT relation

- The systems A and B are initially in thermal equilibrium, at temperatures T_A, T_B.
- They are kept in contact with each other for a time duration τ via an interaction characterized by $H_{\rm int}(\xi_A, \xi_B)$.
- The interaction is switched 'on' at time $t = 0$, and turned 'off' at $t = \tau$.

Phase-space trajectory for forward/backward in time

Statistical description of the arrow of time of a process

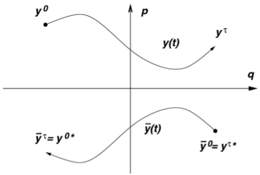


FIG. 1. Twin trajectories $y(t)$ and $\bar{y}(t) = y^*(\tau - t)$ related by time reversal.

- Phase space trajectory is denoted by the variable $\mathbf{y}$
- It is assumed that the phase-space evolution is time-reversal symmetric $\Rightarrow$ for every legitimate forward trajectory $\mathbf{y}^0$ to $\mathbf{y}^\tau$, there exists a time-reversed trajectory $\bar{\mathbf{y}}^0 = \mathbf{y}^{\tau *}$ where $\bar{\mathbf{y}}^\tau = \mathbf{y}^{0*}$.

Classical phase-space description for Jarzynski-Wójcik XFT relation

- Net energy change $\Delta\, E_A$, $\Delta\, E_B$ during the interaction represents the amount of heat transferred i.e., $\mathcal{Q}(\mathbf{y}) = \Delta\, E_B \approx -\Delta\, E_A$.
- The probabilities of heat-exchange $p_\tau(\mathcal{Q})$, $p_\tau(-\mathcal{Q})$ obey the Jarzynski-Wójcik heat exchange fluctuation theorem in the classical scenario:

$$\frac{p_\tau(\mathcal{Q})}{p_\tau(-\mathcal{Q})} = e^{\Delta\beta\,\mathcal{Q}}, \quad \triangle\beta = \frac{1}{k\,T_A} - \frac{1}{k\,T_B}.$$

Jarzynski-Wóczik (JW) heat XFT in the quantum realm (double projection measurement approach)

Enter quantum mechanics

- JW considered two discrete level systems (in thermal equilibrium at temperatures T_A, T_B) and followed the following steps:
  - measure energies E_i^A, E_i^B of the systems initially;
  - allow them to interact *weakly* for a time duration τ;
  - the interaction is turned off;
  - energies E_f^A, E_f^B of both the systems are measured.

Double projection measurement method

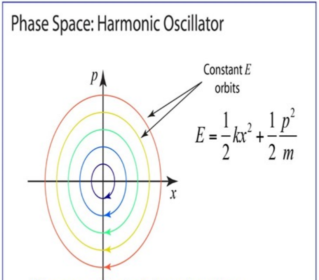
- Energy conservation (interaction between systems is weak)$\Rightarrow$ $E_i^A + E_i^B \approx E_f^A + E_f^B$.
- Heat transfer:

$$\mathcal{Q}_{i\to f} = E_i^B - E_f^B \approx E_f^A - E_i^A$$

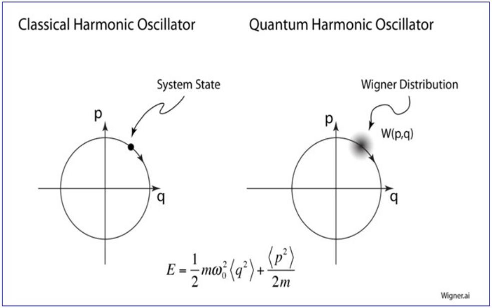
- JW heat XFT in the quantum scenario:

$$\boxed{\ln\left[\frac{p\Big(|i\rangle \xrightarrow{\tau} |f\rangle\Big)}{p\Big(|f\rangle \xrightarrow{-\tau} |i\rangle\Big)}\right] = \triangle\beta\,\mathcal{Q}_{i\to f}.}$$

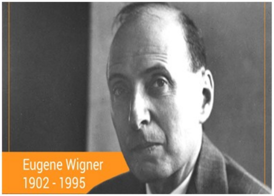
Quantum phase space trajectories?



Phase space of the simple harmonic oscillator. The "orbits" have constant energy.



Wigner Function



JUNE 1, 1932 PHYSICAL REVIEW VOLUME 40

On the Quantum Correction For Thermodynamic Equilibrium

By E. WIGNER
Department of Physics, Princeton University
(Received March 14, 1932)

The probability of a configuration is given in classical theory by the Boltzmann formula exp $[-V/hT]$ where V is the potential energy of this configuration. For high temperatures this of course also holds in quantum theory. For lower temperatures, however, a correction term has to be introduced, which can be developed into a power series of h. The formula is developed for this correction by means of a probability function and the result discussed.

Wigner Distribution Function

position, momentum, wavefunction, Fourier transform

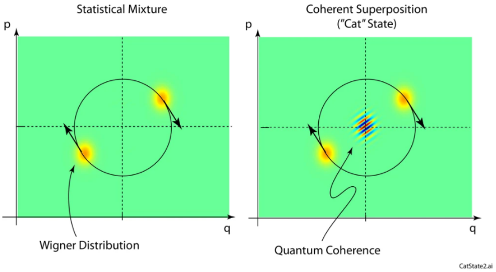
$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \psi^*\left(x-\frac{y}{2}\right)\psi\left(x+\frac{y}{2}\right)e^{-ipy/\hbar}dy$$

Wigner.ai

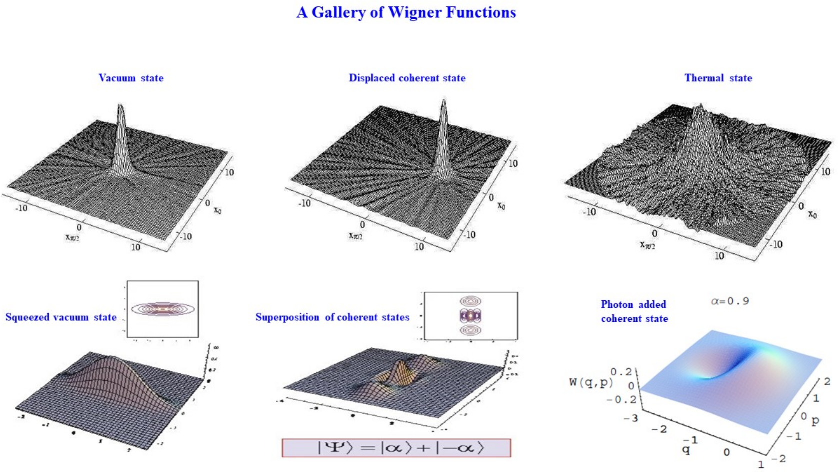
Wigner Function

Expectation value of any operator $\hat{A}(\hat{q},\hat{p}) \longrightarrow A(q,p)$:

$$\langle \hat{A}(\hat{q},\hat{p})\rangle = \mathrm{Tr}\hat{\rho}\, A = \int\int dq\, dp\, W(q,p)\, A(q,p).$$



Wigner Function



Wigner Function

- Characteristic function $\Phi(u,v) = \mathrm{Tr}\left(\hat{\rho}\, e^{i\,(\hat{q}\,u + \hat{p}\,v)}\right)$ of quantum Gaussian state is Gaussian and hence Wigner function is also a Gaussian (*positive*)
(eg., coherent states, thermal states)

Wigner Function of thermal state

- Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}\, m\omega^2\, \hat{q}^2$
- Density operator of thermal state:

$$\begin{aligned}
\hat{\rho} &= \frac{e^{-\beta\,\hat{H}}}{Z(\beta)}, \quad \beta = (k_B\,T)^{-1} \\
Z(\beta) &= \mathrm{Tr}\left(e^{-\beta\,\hat{H}}\right) \\
&= e^{-\beta\hbar\omega/2}(1 - e^{-\beta\hbar\omega})^{-1}.
\end{aligned}$$

Wigner Function of thermal state

- Wigner function of thermal state:

$$
\begin{aligned}
W(q,p) &= \frac{1}{\pi\hbar}\tanh\left(\frac{\beta\hbar\omega}{2}\right)\exp\left[-\frac{2}{\hbar\omega}\tanh\left(\frac{\beta\hbar\omega}{2}\right)H(q,p)\right] \\
&= \frac{1}{2\pi\,\nu_T}\exp\left[-\frac{H(q,p)}{2\,\hbar\omega\,\nu_T}\right]
\end{aligned}
$$

- Here $\nu_T = \frac{1}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right) \longrightarrow$ symplectic eigenvalue.
- Internal energy $U = \hbar\omega\,\nu_T$.

Our Approach

- Wigner distribution formalism has played an important role in developing quantum phase-space formalism involving non-commuting canonical observables $\hat{q}, \hat{p}$.
- It allows one to explore the connection between quantum and classical formalisms.
- Our interest here is to derive the analogue of JW heat-exchange fluctuation relation describing heat transfer processes in the forward and the time-reversed dynamics of two harmonic oscillators A, B (in thermal equilibrium at temperatures T_A, T_B) using the Wigner distribution function formalism.

JW XFT: Wigner function formalism

- The Wigner function $W(\xi^0)$ at time $t = 0$ of the two-mode Gaussian thermal state $\hat{\rho}^0_{AB} = \hat{\rho}^0_{T_A} \otimes \hat{\rho}^0_{T_B}$:

$$W(\xi^0) = \frac{1}{(2\pi)^2\,\nu_{T_A}\nu_{T_B}} \exp\left[-\left(\frac{H_A(\xi^0_A)}{\hbar\omega_A\nu_{T_A}} + \frac{H_B(\xi^0_B)}{\hbar\omega_B\nu_{T_B}}\right)\right]$$

- Here $\xi^0 = (\xi^0_A, \xi^0_B)^T$, $\xi^0_A = (q^0_A, p^0_A)^T$, $\xi^0_B = (q^0_B, p^0_B)^T$ are classical phase-space columns at $t = 0$ and

$$\nu_{T_{A,B}} = \frac{1}{2}\coth\left(\frac{\hbar\omega_{A,B}}{2\,k\,T_{A,B}}\right).$$

JW XFT: Wigner function formalism

- Unitary evolution

$$\hat{U}(\hat{\xi}^{\tau}) = \exp\left[-\frac{i\,\tau}{\hbar}\left(\hat{H}_A(\hat{\xi}^{\tau}_A) + \hat{H}_B(\hat{\xi}^{\tau}_B) + \hat{H}_{\rm int}(\hat{\xi}^{\tau})\right)\right]$$

leads to

$$\rho^0_{AB} \longrightarrow \rho^{\tau}_{AB} \;=\; \hat{U}(\hat{\xi}^{\tau})\,\rho^0_{AB}\,\hat{U}^{\dagger}(\hat{\xi}^{\tau}).$$

- Consequent transformation on the Wigner function:

$$W(\xi^0) \;\longrightarrow\; W(\xi^{\tau}).$$

JW XFT: Wigner function formalism

- One thus obtains

$$W(\xi^\tau) = \frac{1}{(2\pi)^2\,\nu_{T_A}\nu_{T_B}}\exp\left[-\left(\frac{H_A(\xi^\tau_A)}{\hbar\omega_A\,\nu_{T_A}} + \frac{H_B(\xi^\tau_B)}{\hbar\omega_B\,\nu_{T_B}}\right)\right]$$

JW XFT: Wigner function formalism

- Heat probability distribution $p_\tau(\mathcal{Q})$ in terms of the Wigner function:

$$
\begin{aligned}
p_\tau(\mathcal{Q}) &= \int \mathrm{d}\xi^0\, W(\xi^0)\, \delta(\mathcal{Q} - \mathcal{Q}(\xi^0)) \\
&= e^{\triangle\beta_\omega\, \mathcal{Q}} \int \mathrm{d}\bar{\xi}^0\, W(\bar{\xi}^0)\, \delta(\mathcal{Q} + \mathcal{Q}(\bar{\xi}^0)) \\
&= e^{\triangle\beta_\omega\, \mathcal{Q}}\, p_\tau(-\mathcal{Q}).
\end{aligned}
$$

JW XFT: Wigner function formalism

$$
\begin{aligned}
\triangle\beta_\omega &= \beta_{\mathrm{B}\omega} - \beta_{\mathrm{A}\omega} \\
&= \frac{1}{\hbar\omega_{\mathrm{B}}\nu_{T_{\mathrm{B}}}} - \frac{1}{\hbar\omega_{\mathrm{A}}\nu_{T_{\mathrm{A}}}} \\
&= \frac{2\,\tanh\left(\frac{\hbar\omega_{\mathrm{B}}}{2kT_{\mathrm{B}}}\right)}{\hbar\omega_{\mathrm{B}}} - \frac{2\,\tanh\left(\frac{\hbar\omega_{\mathrm{A}}}{2kT_{\mathrm{A}}}\right)}{\hbar\omega_{\mathrm{A}}}.
\end{aligned}
$$

JW-*like* Heat Exchange Fluctuation Relation

JW Heat Exchange-Fluctuation Relation
- 2 time measurement approach

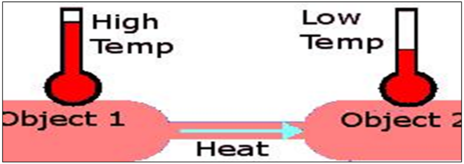
$$\ln\frac{p_\tau(+Q)}{p_\tau(-Q)} = \Delta\beta Q$$

$$\Delta\beta = T_B^{-1} - T_A^{-1}$$

JW -*like* Heat Exchange-Fluctuation Relation
- Wigner function approach

$$\ln\left(\frac{p_\tau(\mathcal{Q})}{p_\tau(-\mathcal{Q})}\right) = \Delta\beta_\omega\, \mathcal{Q}$$

$$\begin{aligned}\Delta\beta_\omega &= \beta_{B\omega} - \beta_{A\omega} \\ &= \frac{1}{\hbar\omega_B\,\nu_{T_B}} - \frac{1}{\hbar\omega_A\,\nu_{T_A}} \\ &= \frac{2\,\tanh\left(\frac{\hbar\omega_B}{2kT_B}\right)}{\hbar\omega_B} - \frac{2\,\tanh\left(\frac{\hbar\omega_A}{2kT_A}\right)}{\hbar\omega_A}.\end{aligned}$$



JW XFT: Wigner function approach

- In the high temperature limit $\frac{\hbar\omega}{kT} \to 0$ we get $\triangle\beta_\omega \to \triangle\beta$.
- The JW XFT like heat exchange fluctuation relation reduces to its classical analogue in this limit.

Wigner function approach

PHYSICAL REVIEW E **100**, 062119 (2019)

Computing characteristic functions of quantum work in phase space

Yixiao Qian and Fei Liu*
School of Physics, Beihang University, Beijing 100191, China

(Received 12 August 2019; revised manuscript received 3 November 2019; published 16 December 2019)

In phase space, we analytically obtain the characteristic functions (CFs) of a forced harmonic oscillator [Talkner *et al.*, Phys. Rev. E **75**, 050102(R) (2007)], a time-dependent mass and frequency harmonic oscillator [Deffner and Lutz, Phys. Rev. E **77**, 021128 (2008)], and coupled harmonic oscillators under driving forces in a simple and unified way. For general quantum systems, a numerical method that approximates the CFs to $\hbar^2$ order is proposed. We exemplify the method with a time-dependent frequency harmonic oscillator and a family of quantum systems with time-dependent even power-law potentials.

DOI: 10.1103/PhysRevE.100.062119

PAPER

Semiclassical work and quantum work identities in Weyl representation

O Brodier[4,1], K Mallick[2] and A M Ozorio de Almeida[3]

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Journal of Physics A: Mathematical and Theoretical, Volume 53, Number 32
Citation O Brodier *et al* 2020 *J. Phys. A: Math. Theor.* **53** 325001
DOI 10.1088/1751-8121/ab8110

References

+ Article and author information

Abstract

We derive a semiclassical nonequilibrium work identity by applying the Wigner–Weyl quantization scheme to the Jarzynski identity for a classical Hamiltonian. This allows us to extend the concept of work to the leading order in $\hbar$. We propose a geometric interpretation of this semiclassical Jarzynski relation in terms of trajectories in a complex phase space and illustrate it with the exactly solvable case of the quantum harmonic oscillator.

- It is for the first time that we have derived a Jarzynski-Wózcik *like* heat exchange fluctuation relation for a system of two quantum harmonic oscillators.

Discussions: Quantum phase-space trajectories??

- The phase-space trajectory concept of the Wigner-Weyl formalism gets hindered by the underlying uncertainty relation.
- However, it gets validated in the classical limit $\hbar \to 0$.
- This explains the reduction of Jarzynski-Wózcik heat exchange fluctuation relation to its classical analogue in the limit $\hbar \to 0$ where it is possible to have a *legitimate* interpretation for the phase-space trajectories.

Discussions: Equipartition theorem

- Equipartition theorem plays a fundamental role in classical statistical physics.
- Equipartition theorem of energy holds universally in classical statistical physics as it neither depends on the number of particles in the ensemble nor on the nature of the potential acting on the particles.

Discussions: Equipartition theorem

- For a system of one dimensional classical harmonic oscillators, in thermal equilibrium at temperature T, contribution to the average energy comes from mean kinetic energy and mean potential energy i.e., $\langle E \rangle = k\,T$.
- In the quantum scenario the average energies depend on frequencies ω_A, ω_B – indicative of the nature of the potential.

Discussions: Equipartition theorem

- Recent papers on equipartition theorem in the quantum realm:

SCIENTIFIC REPORTS

OPEN

Partition of energy for a dissipative quantum oscillator

P. Bialas[1], J. Spiechowicz[1,2] & J. Łuczka[1]

Received: 27 June 2018
Accepted: 18 October 2018
Published online: 31 October 2018

We reveal a new face of the old clichéd system: a dissipative quantum harmonic oscillator. We formulate and study a quantum counterpart of the energy equipartition theorem satisfied for classical systems. Both mean kinetic energy E_k and mean potential energy E_p of the oscillator are expressed as $E_k = \langle \varepsilon_k \rangle$ and $E_p = \langle \varepsilon_p \rangle$, where $\langle \varepsilon_k \rangle$ and $\langle \varepsilon_p \rangle$ are mean kinetic and potential energies per one degree of freedom of the thermostat which consists of harmonic oscillators too. The symbol $\langle \ldots \rangle$ denotes two-fold averaging: (i) over the Gibbs canonical state for the thermostat and (ii) over thermostat oscillators frequencies ω which contribute to E_k and E_p according to the probability distribution $\mathbb{P}_k(\omega)$ and $\mathbb{P}_p(\omega)$, respectively. The role of the system-thermostat coupling strength and the memory time is analysed for the exponentially decaying memory function (Drude dissipation mechanism) and the algebraically decaying damping kernel.

Journal of Physics A: Mathematical and Theoretical

LETTER

Quantum analogue of energy equipartition theorem

P Bialas[1], J Spiechowicz[1] and J Łuczka[1]

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Journal of Physics A: Mathematical and Theoretical, Volume 52, Number 15
Citation P Bialas *et al* 2019 *J. Phys. A: Math. Theor.* 52 15LT01
DOI 10.1088/1751-8121/ab03f2

Journal of Statistical Physics (2020) 179:839–845
https://doi.org/10.1007/s10955-020-02557-5

Check for updates

Quantum Counterpart of Classical Equipartition of Energy

Jerzy Łuczka[1,2]

Published online: 16 May 2020

Abstract
It is shown that the recently proposed quantum analogue of classical energy equipartition theorem for two paradigmatic, exactly solved models (i.e., a free Brownian particle and a dissipative harmonic oscillator) also holds true for all quantum systems which are composed of an arbitrary number of non-interacting or interacting particles, subjected to any confining potentials and coupled to thermostat with arbitrary coupling strength.

Keywords Quantum systems · Equipartition of energy · Quantum analogue

Discussions: Moment generation function for heat-exchange

- The moment generating function of heat distribution $p_\tau(\mathcal{Q})$ in the Wigner-Weyl formalism and the Rényi divergence of order-s:

$$\begin{aligned}
\left\langle e^{-s\,\triangle\beta_\omega\,\mathcal{Q}}\right\rangle &= G_\tau(\triangle\beta_\omega;s) = \int \mathrm{d}\mathcal{Q}\,p_\tau(\mathcal{Q})\,e^{-s\,\triangle\beta_\omega\,\mathcal{Q}} \\
&= \int \mathrm{d}\xi^0\,W(\xi^0)\left\{\int \mathrm{d}\mathcal{Q}\,e^{-s\,\triangle\beta_\omega\,\mathcal{Q}}\delta(\mathcal{Q}-\mathcal{Q}(\xi^0))\right\} \\
&= \int \mathrm{d}\xi^0\,W(\xi^0)\,e^{-s\,\triangle\beta_\omega\,\mathcal{Q}(\xi^0)} \\
&= \int \mathrm{d}\xi^0\,\left[W(\xi^0)\right]^{1-s}\,\left[W(\xi^\tau)\right]^s \\
&= \exp\left[(1-s)\,R_s\left(W^\tau||W^0\right)\right],
\end{aligned}$$

Rényi divergence

- Note:

$$R_s\left(W^0 || W^\tau\right) = \frac{1}{1-s}\,\ln\left\{\int \mathrm{d}\xi^0\,\left[W(\xi^{\,0})\right]^{1-s}\,\left[W(\xi^{\,\tau})\right]^s\right\}$$

denotes the order-s Rényi divergence between the Wigner functions $W(\xi^{\,0})$ and $W(\xi^{\,\tau})$.

Moment generation function for heat-exchange (double projective measurements)

- It has been shown (Wei B B 2018 Relations between heat exchange and Rényi divergences *Phys. Rev.* E **97**, 042107) that heat exchange moment generating function and the order-s Rényi divergences

$$R_s\left(\rho^0_{AB}||\rho^\tau_{AB}\right) = \frac{1}{1-s}\,\ln\left\{\mathrm{Tr}[(\rho^\tau_{AB})^{1-s}\,\left(\rho^0_{AB}\right)^s]\right\}$$

between the initial, final density operators ρ^0_{AB}, ρ^τ_{AB} are related:

$$G_\tau(\triangle\beta; s) = \int \mathrm{d}\mathcal{Q}\, p_\tau(\mathcal{Q})\, e^{-s\,\triangle\beta\,\mathcal{Q}} = \exp\left[(1-s)\,R_s\left(\rho^0_{AB}||\rho^\tau_{AB}\right)\right]$$

Moment generation function for heat-exchange

- In Wei's theoretical derivation the double projection measurement approach has been employed.

Moment generation function for heat-exchange (quantum)

- How different are $R_s\left(\rho^0_{AB}||\rho^\tau_{AB}\right)$ (derived using double projective measurement approach) and $R_s\left(W^0||W^\tau\right)$ (the Wigner phase-space formalism).
- Such study would shine light on the deviation of the double projective measurement method and the Wigner-Weyl phase-space approach.

Relative entropy of two gaussian states

- A formula for Petz-Rényi (quantum) relative entropy of two gaussian states ρ, σ in the boson Fock space $\Gamma(\mathbb{C}^n)$ has been derived:
  K. R. Parthasarathy, *A pedagogical note on the computation of relative entropy of two n-mode gaussian states*, Infinite Dimensional Analysis, Quantum Probability and Applications ICQPRT 2021, **390**, pp 55-72 (2022).

Experimental measurement of Wigner function of atoms in optical traps

IOP Publishing
Journal of Physics B: Atomic, Molecular and Optical Physics
J. Phys. B: At. Mol. Opt. Phys. 55 (2022) 194004 (10pp)
https://doi.org/10.1088/1361-6455/ac8bb8

Direct measurement of the Wigner function of atoms in an optical trap

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[2] Quantum Engineering Technology Laboratories, H. H. Wills Physics Laboratory and Department of Electrical and Electronic Engineering, University of Bristol, Bristol BS8 1FD, United Kingdom

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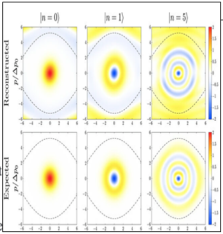
Abstract

We present a scheme to directly probe the Wigner function of the motional state of a neutral atom confined in an optical trap. The proposed scheme relies on the well-established fact that the Wigner function at a given point (x, p) in phase space is proportional to the expectation value of the parity operator relative to that point. In this work, we show that the expectation value of the parity operator can be directly measured using two auxiliary internal states of the atom: parity-even and parity-odd motional states are mapped to the two internal states of the atom through a Ramsey interferometry scheme. The Wigner function can thus be measured point-by-point in phase space with a single, direct measurement of the internal state population. Numerical simulations show that the scheme is robust in that it applies not only to deep, harmonic potentials but also to shallower, anharmonic traps.

nature physics

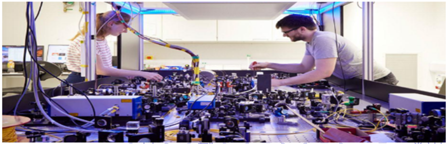
Article
https://doi.org/10.1038/s41567-022-01890-8

Time-of-flight quantum tomography of an atom in an optical tweezer



M. O. Brown[1,2], S. R. Muleady[1,2,3], W. J. Dworschack[1,2], R. J. Lewis-Swan[4,5], A. M. Rey[1,2,3], O. Romero-Isart[6,7] & C. A. Regal[1,2]

A single particle trapped in a harmonic potential can exhibit rich motional quantum states within its high-dimensional state space. Quantum characterization of motion is key, for example, in controlling or harnessing motion in trapped ion and atom systems or observing the quantum nature of the vibrational excitations of solid-state objects. Here we show that the direct measurement of position and momentum can be used for quantum tomography of motional states of a single trapped particle. We obtain the momentum of an atom in an optical tweezer via time-of-flight measurements, which, combined with trap harmonic evolution, grants us access to all quadrature distributions. Starting with non-classical motional states of a trapped neutral atom, we demonstrate the Wigner function negativity and coherence of non-stationary states. Our work will enable the characterization of the complex neutral atom motion that is of interest for quantum information and metrology, and for investigations of the quantum behaviour of massive levitated particles.



Discussions: Correlated objects & JW XFT

PHYSICAL REVIEW E **92**, 042113 (2015)

Exchange fluctuation theorem for correlated quantum systems

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(Received 17 May 2012; revised manuscript received 3 July 2015; published 6 October 2015)

We extend the exchange fluctuation theorem for energy exchange between thermal quantum systems beyond the assumption of molecular chaos, and describe the nonequilibrium exchange dynamics of correlated quantum states. The relation quantifies how the tendency for systems to equilibrate is modified in high-correlation environments. In addition, a more abstract approach leads us to a "correlation fluctuation theorem". Our results elucidate the role of measurement disturbance for such scenarios. We show a simple application by finding a semiclassical maximum work theorem in the presence of correlations. We also present a toy example of qubit-qudit heat exchange, and find that non-classical behaviour such as deterministic energy transfer and anomalous heat flow are reflected in our exchange fluctuation theorem.

→ "Correlations result in a modification of the XFT relation and can enhance the probability of heat flowing in the backward direction"

Discussions: Experimental resusts on JW XFT

PHYSICAL REVIEW A **100**, 042119 (2019)

Experimental demonstration of the validity of the quantum heat-exchange fluctuation relation in an NMR setup

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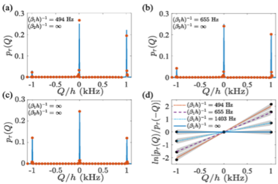
(Received 27 November 2018; revised manuscript received 4 August 2019; published 24 October 2019)

We experimentally explore the validity of the Jarzynski and Wójcik quantum heat-exchange fluctuation relation by implementing an interferometric technique in liquid-state nuclear magnetic resonance setup and study the heat-exchange statistics between two coupled spin-1/2 quantum systems. We experimentally emulate two models—(i) the *XY*-coupling model, containing an energy conserving interaction between the qubits, and (ii) the *XX*-coupling model—and analyze the regimes of validity and violation of the fluctuation symmetry when the composite system is prepared in an uncorrelated initial state with individual spins prepared in local Gibbs thermal states at different temperatures. We further extend our analysis for heat exchange by incorporating correlation in the initial state. We support our experimental findings by providing exact analytical results. Our experimental approach is general and can be systematically extended to study heat statistics for more complex out-of-equilibrium many-body quantum systems.

→ "Inclusion of any finite amount of correlation in the initial state also leads to a break down of the fluctuation symmetry and, interestingly, reverses the direction of the heat flow against thetemperature bias, thereby providing an additional knob for controlling heat flow"



(e)

$(\beta_1 h)^{-1}$(Hz)	Theory $\Delta\beta$ (kHz)$^{-1}$	Experiment $\Delta\beta$ (kHz)$^{-1}$
494	2.02	2.15 ± 0.08
655	1.52	1.55 ± 0.03
1403	0.71	0.70 ± 0.01

FIG. 4. (a–c) PDF of heat exchange for the *XY* model for different spin temperatures of F_1: (a) $(\beta_1 h)^{-1} = 494$ Hz, (b) $(\beta_1 h)^{-1} = 655$ Hz, and (c) $(\beta_1 h)^{-1} = \infty$. $(\beta_2 h)^{-1} = \infty$. Solid blue lines and red dots correspond to theoretical and experimental results, respectively. (d) Verification of Jarzynski and Wójcik heat XFT plots for $\ln[p_\tau(Q)/p_\tau(-Q)]$ as a function of Q/h for four F_1 temperatures. Shaded regions indicate the simulated 5% pulse errors in the experiment. (e) Table listing theoretical and experimentally obtained values for the slope $\Delta\beta = \beta_1 - \beta_2$, $\beta_i = 1/k_B T_i$ from (d). All other parameters are the same as in Fig. 3.

Discussions: Beyond double projective measurements

Quantum Fluctuation Theorems beyond Two-Point Measurements

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(Received 9 September 2019; accepted 7 February 2020; published 2 March 2020)

We derive detailed and integral quantum fluctuation theorems for heat exchange in a quantum correlated bipartite thermal system using the framework of dynamic Bayesian networks. Contrary to the usual two-projective-measurement scheme that is known to destroy quantum features, these fluctuation relations fully capture quantum correlations and quantum coherence at arbitrary times. We further obtain individual integral fluctuation theorems for classical and quantum correlations, as well as for local and global quantum coherences.

DOI: 10.1103/PhysRevLett.124.090602

PHYSICAL REVIEW LETTERS **127,** 180603 (2021)

Experimental Validation of Fully Quantum Fluctuation Theorems Using Dynamic Bayesian Networks

Kaonan Micadei,[1] John P. S. Peterson,[2,3] Alexandre M. Souza,[2] Roberto S. Sarthour,[2] Ivan S. Oliveira,[2] Gabriel T. Landi,[4] Roberto M. Serra,[5,6] and Eric Lutz[1]
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(Received 3 February 2021; revised 10 March 2021; accepted 5 October 2021; published 29 October 2021)

Fluctuation theorems are fundamental extensions of the second law of thermodynamics for small systems. Their general validity arbitrarily far from equilibrium makes them invaluable in nonequilibrium physics. So far, experimental studies of quantum fluctuation relations do not account for quantum correlations and quantum coherence, two essential quantum properties. We here apply a novel dynamic Bayesian network approach to experimentally test detailed and integral fully quantum fluctuation theorems for heat exchange between two quantum-correlated thermal spins-1/2 in a nuclear magnetic resonance setup. We concretely verify individual integral fluctuation relations for quantum correlations and quantum coherence, as well as for the sum of all quantum contributions. We further investigate the thermodynamic cost of creating correlations and coherence.

"Due to its inherent projective nature, double projective measurement approach fails to quantify the thermodynamic effects of quantum correlations and quantum coherence that may be present in initial and final states. Since these are two central quantum features that affect the expression of the second law, such fluctuation theorems may be viewed as not fully quantum"

- A trajectory based approach employing dynamic Baysean networks → derivation of a *fully* quantum fluctuation theorem for heat exchange in a correlated bipartite thermal system.

Other approaches on JW XFT

PHYSICAL REVIEW E **98**, 052106 (2018)

Heat distribution of a quantum harmonic oscillator

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(Received 13 July 2018; published 6 November 2018)

We consider a thermal quantum harmonic oscillator weakly coupled to a heat bath at a different temperature. We analytically study the quantum heat exchange statistics between the two systems using the quantum-optical master equation. We exactly compute the characteristic function of the heat distribution and show that it verifies the Jarzynski-Wójcik fluctuation theorem. We further evaluate the heat probability density in the limit of long thermalization times, both in the low- and the high-temperature regimes, and investigate its time evolution by calculating its first two cumulants.

DOI: 10.1103/PhysRevE.98.052106

$$-\frac{d\rho(t)}{dt} = i\omega[a^\dagger a, \rho] + \frac{\gamma}{2}\bar{n}_2(aa^\dagger\rho + \rho a a^\dagger - 2a^\dagger\rho a) + \frac{\gamma}{2}(\bar{n}_2 + 1)(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger),$$

$$P(Q) = \frac{1 - e^{-\hbar\omega\beta_1} - e^{-\hbar\omega\beta_2} + e^{-\hbar\omega(\beta_1+\beta_2)}}{1 - e^{-\hbar\omega(\beta_1+\beta_2)}} \times \sum_n \delta(Q - n\hbar\omega) + \delta(Q + n\hbar\omega)\begin{cases} e^{-\beta_2 Q}, & Q \geqslant 0 \\ e^{\beta_1 Q}, & Q < 0. \end{cases}$$

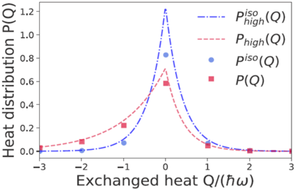

FIG. 1. Asymptotic quantum heat distribution $P(Q)$, Eq. (9), for a harmonic oscillator at inverse temperature β_1 weakly coupled to a bath at inverse temperature β_2 (red squares), compared with the symmetric isothermal heat distribution $P^{\text{iso}}(Q)$, Eq. (11), obtained for $\beta = \beta_1 = \beta_2$ (blue dots). The respective blue dot-dashed and red dashed lines represent the corresponding classical heat distribution given by Eq. (12). Parameters are $\beta_1 = 1$, $\beta_2 = 2.5$, and $\beta = 2.5$.

- The quantum distribution is discrete with spacing corresponding to the level interval of the harmonic oscillator.
- Narrower than that of the classical distribution.

Thermal relaxation of two harmonic oscillators

PHYSICAL REVIEW A **77**, 032121 (2008)

Relaxation phenomena in a system of two harmonic oscillators

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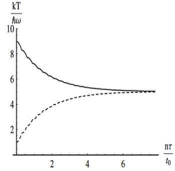
(Received 29 May 2007; published 28 March 2008)

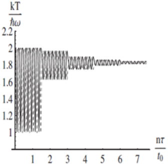
We study the process by which quantum correlations are created when an interaction Hamiltonian is repeatedly applied to a system of two harmonic oscillators for some characteristic time interval. We show that, for the case where the oscillator frequencies are equal, the initial Maxwell-Boltzmann distributions of the uncoupled parts evolve to a new Maxwell-Boltzmann distribution through a series of transient Maxwell-Boltzmann distributions, or quasistationary, nonequilibrium states. Further, we discuss why the equilibrium reached when the two oscillator frequencies are unequal is not a thermal one. All the calculations are exact and the results are obtained through an iterative process, without using perturbation theory.

DOI: 10.1103/PhysRevA.77.032121 PACS number(s): 03.65.Yz, 67.25.du, 31.70.Hq, 67.30.H−

E C G Sudarshan

$$\rho_1(0)\otimes\rho_2(0)\xrightarrow{\hat{U}(\tau)}\mathrm{Tr}_2[\hat{U}\rho_{12}(0)\hat{U}^\dagger]\otimes\mathrm{Tr}_1[\hat{U}\rho_{12}(0)\hat{U}^\dagger]\xrightarrow{\hat{U}(\tau)}\cdots\xrightarrow{\hat{U}(\tau)}\mathrm{Tr}_2[\hat{U}\rho_{12}[(n-1)\tau]\hat{U}^\dagger]\otimes\mathrm{Tr}_1[\hat{U}\rho_{12}[(n-1)\tau]\hat{U}^\dagger],$$





PHYSICAL REVIEW VOLUME 129, NUMBER 4 15 FEBRUARY 1963

Relaxation Phenomena in Spin and Harmonic Oscillator Systems

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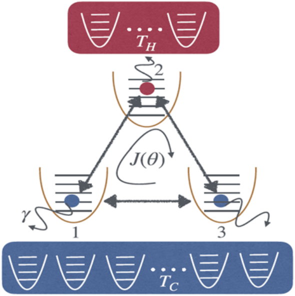
(Received 2 August 1962)

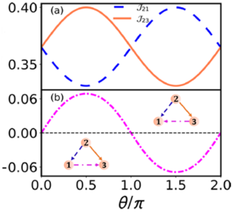
A method is developed for generating relaxation by introducing a fundamental interval τ and a stirring hypothesis. The application to spin and harmonic oscillator systems is discussed in some detail. All the results are obtained by exact calculations without applying perturbation theory as the systems considered are simple and completely soluble. Equations similar to phenomenological Bloch equations are derived in the case of spin systems. The relaxation times obtained by the application of the theory are not only proportional to the strength of interaction, but also to the fundamental interval τ which plays an important role in the theory. It is shown that in the case of a harmonic oscillator system, an initial Boltzmann distribution relaxes to a final equilibrium Boltzman distribution through a sequence of transient Boltzmann distributions.

Quantum control of heat current

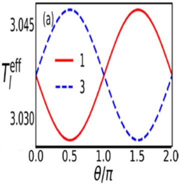
Chakraborty et. al, *Quantum control of heat current*, Phys. Rev. A **110**, 042216 (2024) → TCG Crest



Bosonic trimers driven by two different heat baths



The steady-state non-equilibrium heat currents



Steady-state temperature

$$H = \sum_{l=1}^{3} \omega a_l^\dagger a_l + \sum_{lm} J_{lm}(e^{i\theta_{lm}} a_l^\dagger a_m + e^{-i\theta_{lm}} a_l a_m^\dagger). \quad \boxed{J_{lm} = J} \quad \boxed{\text{cumulative phase parameter } \theta = \theta_{12} + \theta_{23} + \theta_{31}}$$

Funding

- Department of Science and Technology(DST), India (DST/ICPS/QUST/Theme-2/2019/Q107)

A R Usha Devi *et al J. Stat. Mech.* (2021) 023209

Heat exchange and fluctuation in Gaussian thermal states in the quantum realm

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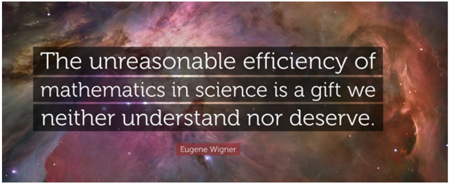
https://doi.org/10.1088/1742-5468/abdd14

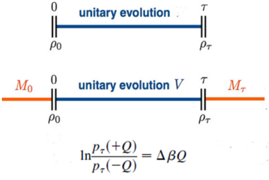
Abstract. The celebrated exchange fluctuation theorem—proposed by Jarzynski and Wójcik (2004 *Phys. Rev. Lett.* **92** 230602) for heat exchange between two systems in thermal equilibrium at different temperatures—is explored here for quantum Gaussian states in thermal equilibrium. We employ the Wigner distribution function formalism for quantum states, which exhibits a close resemblance to the classical phase-space trajectory description, to arrive at a *formal* Jarzynski–Wójcik result. For two Gaussian states in thermal equilibrium at two different temperatures kept in contact with each other for a fixed duration of time, we show that the Jarzynski–Wójcik-*like* relation reduces to the corresponding classical result in the limit $\hbar \to 0$.

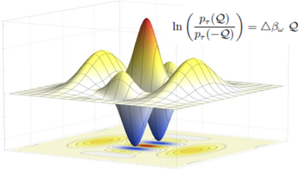
Sudha

A K Rajagopal

A M Jayannavar
(22 July 1956 - 22 November 2021)







As Science has become more abstract and remote from everyday experience, the role of metaphor in our descriptions of the world has become more central. The language that nature speaks, as Galileo long pointed out, is mathematics. The language that ordinary human beings speak .... is metaphor. Lightman ends his discussion with another metaphor: "*We are blind men, imagining what we don't see.*"

- Freeman Dyson, The Scientist as Rebel

Thanks for your kind attention