Repeater-based optimal quantum teleportation in noisy scenario: the case of qubits

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Motivation

- Given any two-qudit state ρ_{AB}, shared between Alice (sender) and Bob (receiver), the optimal teleportation fidelity $\mathcal{F}_\rho$ is known to be dependent on the optimal singlet fraction F_ρ of ρ as: $\mathcal{F}_\rho = (dF_\rho + 1)/(d+1)$.
- The optimal teleportation protocol uses measurement in (generalized) Bell basis.
- Does that mean that in a network scenario – where Alice and Bob$_1$ share a noisy two-qudit state ρ_1, Bob$_1$ and Bob$_2$ share a noisy two-qudit state ρ_2, ..., Bob$_n$ and Charlie share a noisy two-qudit state ρ_{n+1} – the same optimal scheme would provide the best approach for establishing the optimal teleportation channel between Alice and Charlie?
- Here we address this question for $d = 2$ case, and our answer differs (in general) from the above-mentioned paradigm.

Plan of the talk

1. Quantum Teleportation
2. Optimal Teleportation with two-qubit state
3. Optimal distribution of Teleportation channel via LOCC
4. Optimal LOCC protocols for three-node scenario
5. Complexity beyond three-node scenario
6. Maintaining teleportation fidelity with less entanglement
7. Summary

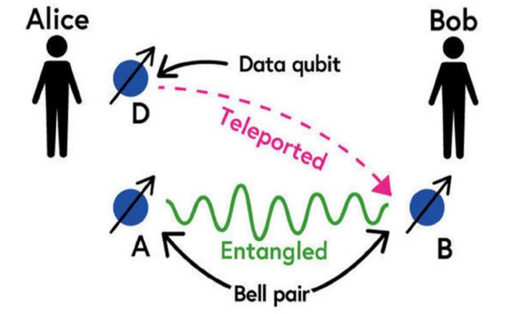
Plan of the talk

Quantum Teleportation: Basic idea

- It is a fundamental protocol to transmit quantum information from a sender (Alice) to the receiver (Bob) using shared entanglement and LOCC.

$$|\psi_{in}\rangle_{A'}\langle\psi_{in}| \otimes \rho_{AB} \longrightarrow \tau_{A'A} \otimes \chi_B^{out}$$

where $|\psi_{in}\rangle\langle\psi_{in}| \implies input\ state, \qquad \chi^{out} \implies output\ state$



Perfect Quantum Teleportation protocol

- Unknown qubit state given to Alice : $|\psi_{in}\rangle = a|0\rangle + b|1\rangle, \quad |a|^2 + |b|^2 = 1$
- Shared entanglement between Alice - Bob : $|\Phi_0\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Bell measurement by Alice	cbits	Teleported state to Bob	Correction by Bob	Output χ^{out}
$\langle\Phi_0| \langle\Phi_0|$	00	$a|0\rangle + b|1\rangle$	$\mathbb{I}_2$	$a|0\rangle + b|1\rangle$
$|\Phi_1\rangle\langle\Phi_1|$	11	$a|0\rangle - b|1\rangle$	σ_3	$a|0\rangle + b|1\rangle$
$|\Phi_2\rangle\langle\Phi_2|$	01	$a|1\rangle + b|0\rangle$	σ_1	$a|0\rangle + b|1\rangle$
$|\Phi_3\rangle\langle\Phi_3|$	10	$a|1\rangle - b|0\rangle$	σ_2	$a|0\rangle + b|1\rangle$

$$|\Phi_{0,1}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \quad \text{and} \quad |\Phi_{2,3}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

Plan of the talk

Quantum Teleportation: Standard figure of merit

- The standard figure of merit for QT ($|\psi\rangle_{A'}\langle\psi| \otimes \rho_{AB} \longrightarrow \tau_{A'A} \otimes \chi_B$) is the average fidelity

$$\langle f_{\rho_{AB}} \rangle = \int d\psi \; \langle \psi | \chi_B | \psi \rangle_{A'}$$

- The average fidelity can be transformed by local unitary (LU) rotations and maximal fidelity given by

$$\mathcal{F}_{\rho_{AB}} = \max_{U_1, U_2} \langle f_{\rho'_{AB}} \rangle$$

where $\rho'_{AB} = (U_1 \otimes U_2) \; \rho_{AB} \; (U_1 \otimes U_2)^\dagger$.

- $F_{\rho_{AB}}$ is known as the maximal fidelity of ρ_{AB} and the protocol $\rho_{AB} \rightarrow \rho'_{AB}$ is known as **optimal** [1] [2] only under LU transformation.

[1] P. Badziag, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 62, 012311 (2000).

[2] R. Horodecki, M. Horodecki, and P. Horodecki, Phys. Lett. A 222, 21 (1996).

Fully Entangled Fraction

- **Definition**: For any bipartite state ρ_{AB}, fully entangled fraction (FEF) is defined as

$$F_{\rho_{AB}} = \max_{|\Phi\rangle} \langle\Phi|\,\rho\,|\Phi\rangle\,, \quad \text{where} \quad |\Phi\rangle \in MES$$

- If $\mathcal{F}_{\rho_{AB}}$ is the maximal fidelity of ρ_{AB}, then FEF of ρ_{AB} can be computed as

$$\mathcal{F}_{\rho_{AB}} = \frac{2\ F_\rho + 1}{3},$$

  where $F_{\rho_{AB}} = \langle\Phi_3|\,\tilde{\rho}_{AB}\,|\Phi_3\rangle$ is nothing but the singlet fraction of $\tilde{\rho}_{AB}$ [3],[4], where $|\Phi_3\rangle$ is the Singlet state.

- A two-qubit state ρ_{AB} is said to be useful under standard protocol *iff* $F_{\rho_{AB}} > \frac{1}{2}$.

[3] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60, 1888 (1999).

[4] P. Badziag, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 62, 012311 (2000).

Maximal fidelity is not LOCC monotone

- The definition of *maximal teleportation fidelity* is valid only under standard protocol (a restricted LOCC).
- In general sender and receiver has freedom of performing any type of LOCC.
- Maximal fidelity can increase under deterministic LOCC and there exist a sufficient protocol for any two-qubit state.

- **Optimal Fully Entangled Fraction:**

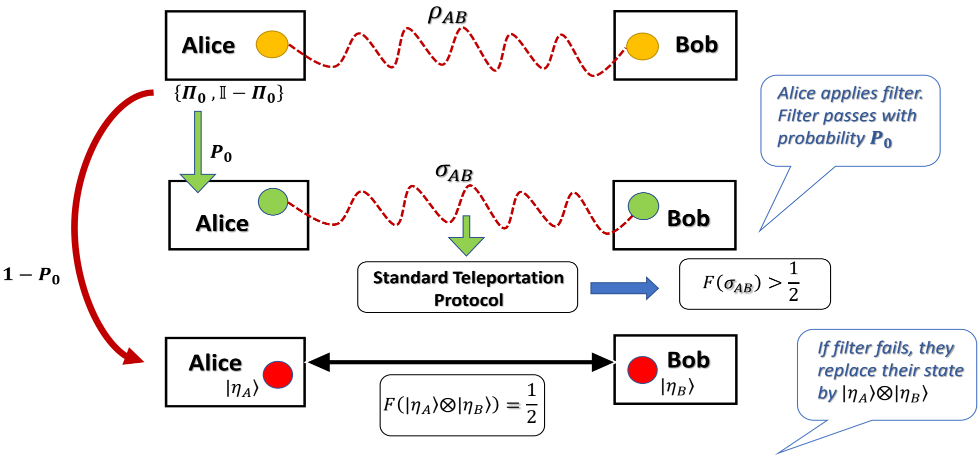
$$F^*(\rho_{AB}) = \max_{\Lambda \in LOCC} F(\Lambda(\rho_{AB}))$$

- **Optimal Teleportation Fidelity:**

$$\mathcal{F}^*(\rho_{AB}) = \frac{2\ F^*(\rho_{AB}) + 1}{3}$$

The Optimal Teleportation protocol

- *The optimal protocol for maximizing teleportation fidelity is a one-way TP LOCC protocol.*



The Optimal Teleportation protocol

- The optimal teleportation fidelity of any two-qubit state can be expressed as [5]

$$F^*(\rho_{AB}) = \max_{A \in SL(2,\mathbb{C})} P_0 \; F(\sigma_{AB}) + \frac{1-P_0}{2}, \quad \text{where} \quad \sigma_{AB} = \frac{(A \otimes \mathbb{I})\; \rho_{AB}\; (A^\dagger \otimes \mathbb{I})}{P_0}$$

- If ρ_{AB} is entangled then $F^*(\rho_{AB}) > \frac{1}{2}$.
- One-way LOCC from Alice to Bob is sufficient to achieve the optimal teleportation fidelity.

[5] F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003), F. Verstraete, J. Dehaene, and B. DeMoor, Phys. Rev. A 64, 010101(R) (2001).

How to compute Optimal Teleportation Fidelity (OTF)

- To compute OTF of ρ_{AB} one needs to find suitable filtering operation (bringing the state to its canonical form ρ^c_{AB}).
- This is not sufficient because one needs to find the quantity, $P\ F(\rho^c_{AB}) + \frac{1-P}{2}$, where $P = Tr\left[\rho_{AB}(A^\dagger A \otimes \mathbb{I})\right]$ is the probability.
- Hence, one needs to solve the following **semi-definite program** (SDP) [6]:

$$F^*(\rho_{AB}) = \ maximize \quad \frac{1}{2} - Tr\left(\rho_{AB}^{\Gamma_B}\ X\right)$$
$$s.t. \quad 0 \le X \le \mathbb{I}_4 \quad and \quad -\frac{\mathbb{I}_4}{2} \le X^{\Gamma_B} \le \frac{\mathbb{I}}{2} \quad and \quad rank(X) = 1$$

[6] F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)

Upper bound of OTF

- The OTF of any two-qubit state is upper bounded as [7]

$$F^*(\rho_{AB}) \leq \frac{1 + N(\rho_{AB})}{2} \leq \frac{1 + C(\rho_{AB})}{2},$$

where $N(\rho_{AB})$ is the negativity and $C(\rho_{AB})$ is the concurrence of ρ_{AB}.

- The upper bound is saturated only for the class of states that satisfy

$$\rho_{AB}^{\Gamma_B} |\Phi\rangle = \lambda_{min} |\Phi\rangle,$$

where $\lambda_{min} < 0$ is the smallest eigenvalue and the eigenvector $|\Phi\rangle$ is a maximally entangled state (MES).

- **Example**: Entangled Bell diagonal states, pure entangled states and so on.

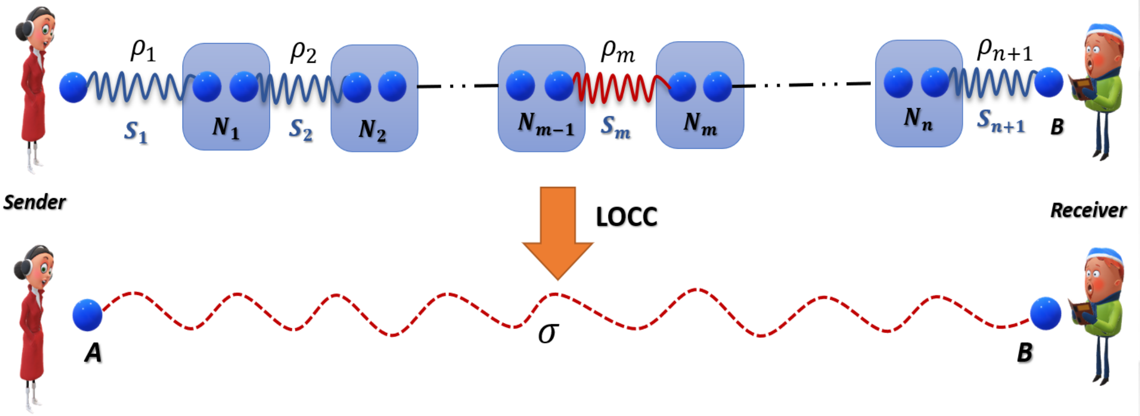
[7] F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)

Plan of the talk

Motivation

- We know the optimal teleportation protocol for a two-qubit state.
- Although the protocol is state dependent but we know one-way LOCC is sufficient.
- **Question**: Can we simply extend such problem over a network?
- Consider a $N-$node scenario where a sender wants to establish optimal qubit teleportation channel with a desired receiver.
- The intermediate nodes act like quantum repeaters where there are single copy of preshared two-qubit states.
- **Question**: How much teleportation fidelity they can establish between Sender and Receiver with LOCC and the given resources ?

Motivation

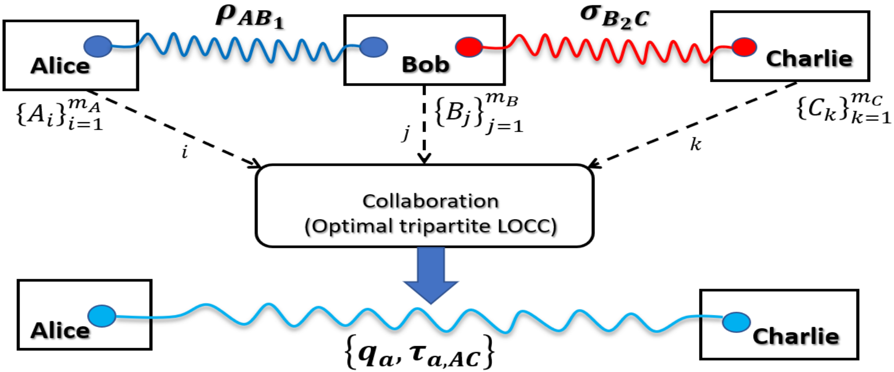


There are many relevant works in this direction.[8]

[8] Nature **605**, pp. 663–668 (2022), IEEE INFOCOM 2022 (doi: 10.1109/INFOCOMWKSHPS54753.2022.9798300), IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS, VOL.38, NO.3, MARCH 2020

Basic three-node scenario

We start with only three nodes, namely, Alice, Bob and Charlie.



$$F^*_{AC}(\rho_{AB_1}, \sigma_{B_2C}) = \max_{\Lambda \in LOCC} F(\Lambda(\rho_{AB_1} \otimes \sigma_{B_2C})) > \frac{1}{2}$$

Objective

Consider three spatially separated parties, namely, Alice ($\mathcal{A}$), Bob ($\mathcal{B}$) and Charlie ($\mathcal{C}$) such that $\mathcal{AB}$ share an arbitrary state ρ_{AB_1} with local dimensions $d_A = d_{B_1} = d \geq 2$ and $\mathcal{BC}$ share another arbitrary state σ_{B_2C} with local dimensions $d_{B_2} = d_C = d \geq 2$. All parties have freedom to perform LOCC. The task is to distribute entanglement between Alice and Charlie such that its fully entangled fraction is maximized over all three-party LOCC and it is strictly greater than the classical bound $\frac{1}{d}$.

How to implement optimal LOCC ?

- In general it is difficult to answer what is optimal implementation.
- Even a finite $n-$round LOCC on a bipartite system is a strict subset of LOCC i.e., $LOCC_N \subset LOCC$.
- Any LOCC can be expressed as a Separable operation (SEP), converse is not true in general because $LOCC \subset SEP$.
- SEP has a simple operator structure and therefore one can say

$$F^*_{SEP}(\rho_{AB_1}, \sigma_{B_2C}) = \max_{|\Phi\rangle \in MES} \max_{\Lambda \in SEP} \langle \Phi_{AC}| Tr_B \left(\Lambda(\eta_{AB_1B_2C})\right) |\Phi_{AC}\rangle$$

$$= \max_{\{m_i,\ N_i\}} \sum_i Tr\left(\tilde{\chi}_{AC}(N_i)\ X_{m_i}\right) \geq F^*_{LOCC}(\rho_{AB_1}, \sigma_{B_2C}).$$

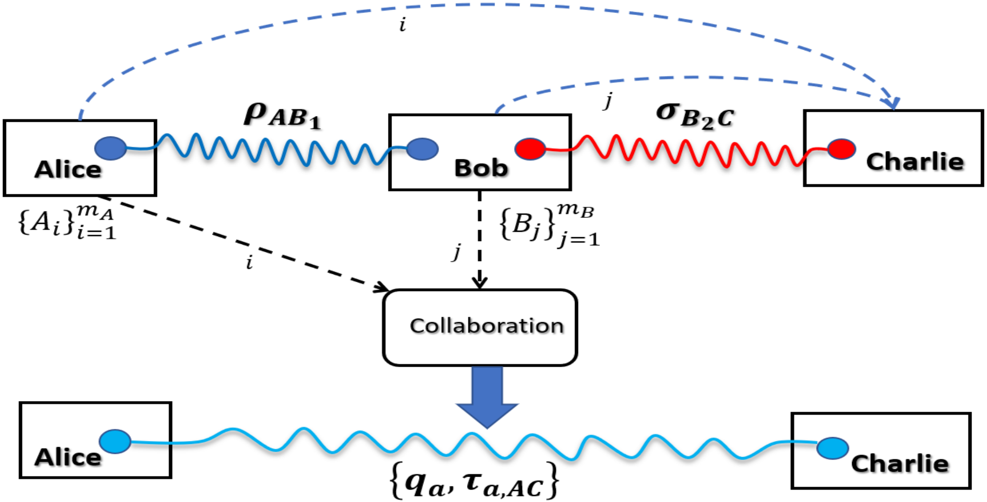
- Detailed calculations are in the Appendix.

- Here $\sqrt{d}\,|\Phi_0\rangle = \sum_{i=0}^{d-1} |ii\rangle$ is the MES in $\mathbb{C}^d \otimes \mathbb{C}^d$
- $X_{m_i} = (m_i^\dagger \otimes \mathbb{I})\; P_{\Phi_0}\; (m_i \otimes \mathbb{I})$ is a rank one projector.
- $\tilde{\chi}_{AC}(N_i) = Tr_B\left((\mathbb{I} \otimes n_i^\dagger n_i \otimes \mathbb{I})\; \eta_{AB_1B_2C}\right)$ where $N_i = n_i^\dagger n_i \geq 0$.
- An important thing to observe is that the minimum value it can take is $\frac{1}{d}$ i.e., the classical bound.
- This is because for a given pair of states $(\rho_{AB_1}, \sigma_{B_2C})$ if the fully entangled fraction value goes below $\frac{1}{d}$, then it is possible that Alice and Charlie replace their existing state by a pure product state using LOCC.

Sufficient protocol

- The sufficient protocol states that a one-way LOCC (LOCC_1) from Alice to Charlie and another LOCC_1 from Bob to Charlie is sufficient.
- However, a both-way LOCC (possibly LOCC_n) is required between Alice and Bob. Although it is difficult to give lower bound on n in general.
- Thus even under separable operations there is a complicated optimization between Alice and Bob.
- For any given states ρ_{AB_1} and σ_{B_2C}, one can propose an optimization problem under three-party SEP operation to obtain $F^*_{SEP}(\rho_{AB_1}, \sigma_{B_2C})$. For details refer to Appendix.

Sufficient protocol



Upper bounds on the optimal teleportation fidelity

- Let us consider a case when $\mathcal{AB}$ share a two-qubit state ρ_{AB_1} and $\mathcal{BC}$ share a two-qubit state σ_{B_2C} *i.e.* d=2.
- From the LOCC argument one can propose the following upper bounds on $F^*_{AC}(\rho,\sigma)$.

$$F^*_{AC}(\rho,\sigma) \leq \min\{F^*_1,\ F^*_2\}, \quad \text{where}$$

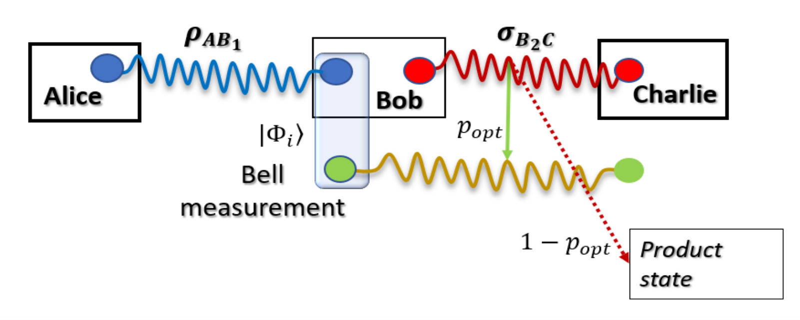
$$F^*_1 = \min\{F^*(\rho_{AB_1}),\ F^*(\sigma_{B_2C})\} \quad \text{and} \quad F^*_2 = \frac{1 + C(\rho_{AB_1})\ C(\sigma_{B_2C})}{2}$$

- Note that there does not exist a general ordering i.e., whether $F^*_1 \geq F^*_2$ holds or vice versa, depends on the chosen pair of states.

Plan of the talk

What intuition would say?

- If we think intuitively, one could say: *Suppose Bob-Charlie share a higher fidelity state i.e.,* $F^*(\rho_{AB_1}) \leq F^*(\sigma_{B_2C})$. *So first make optimal post-processing on* σ_{B_2C}. *Then simply "teleport" one part of* ρ_{AB_1} *using standard teleportation. After that do further post-processing if necessary.*



- Unfortunately this is not the optimal protocol.

Protocols

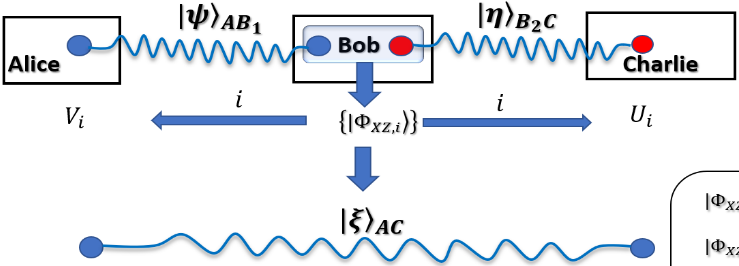
- **Protocol,** $\mathcal{P}_1$: Bob first performs a two-qubit projective measurement in any maximally entangled basis and then classically communicate the outcome to Alice and Charlie. After then, Bob's system is discarded and a one-way LOCC is performed from Alice to Charlie.
- **Protocol,** $\mathcal{P}_2$: Bob first performs a two-qubit projective measurement in a given basis $\{|\eta_i\rangle\}_{i=0}^{3}$ of non-maximally entangled states (excluding all maximally entangled basis) and then communicate the outcome to Alice and Charlie. After then, Bobs system is discarded and a one-way LOCC is performed from Alice or Charlie. The Bob's measurement basis is restricted as

$$
\begin{aligned}
|\eta_0\rangle &= \alpha_0 |00\rangle + \alpha_1 |11\rangle\,, \quad |\eta_1\rangle = \alpha_1 |00\rangle - \alpha_0 |11\rangle\,, \\
|\eta_2\rangle &= \beta_0 |01\rangle + \beta_1 |10\rangle\,, \quad |\eta_1\rangle = \beta_1 |01\rangle - \beta_0 |10\rangle\,,
\end{aligned}
$$

where $\alpha_0 > \alpha_1 > 0$ and $\beta_0 > \beta_1 > 0$.

Class of states for which $F^*_{AC} = F^*_2$ can be achieved

- When both $\mathcal{AB}$ and $\mathcal{BC}$ share a two-qubit pure entangled state.



$$F^{opt}_{AC} = \frac{1 + C(|\Psi_{AB_1}\rangle)C(|\eta_{AB_2}\rangle)}{2}$$

$$|\Phi_{XZ,0}\rangle = \frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle)$$
$$|\Phi_{XZ,1}\rangle = \frac{1}{\sqrt{2}}(|0+\rangle - |1-\rangle)$$
$$|\Phi_{XZ,2}\rangle = \frac{1}{\sqrt{2}}(|0-\rangle + |1+\rangle)$$
$$|\Phi_{XZ,3}\rangle = \frac{1}{\sqrt{2}}(|0-\rangle - |1-\rangle)$$

$$|\Psi_{AB_1}\rangle = \sqrt{a}\,|0_A\rangle \otimes |0_{B_1}\rangle + \sqrt{1-a}\,|1_A\rangle \otimes |1_{B_1}\rangle, \qquad \frac{1}{2} \leq a < 1$$

$$|\eta_{B_2C}\rangle = \sqrt{b}\,|0_{B_2}\rangle \otimes |0_C\rangle + \sqrt{1-b}\,|1_{B_2}\rangle \otimes |1_C\rangle, \qquad \frac{1}{2} \leq b < 1$$

Class of states for which $F^*_{AC} = F^*_1$ can be achieved

- Suppose $\rho_{AB_1} = (1-p)\ket{\psi}\bra{\psi} + p\ \ket{01}\bra{01}$ with $\ket{\psi} = \sqrt{\delta}\ket{00} + \sqrt{1-\delta}\ket{11}$, $0 < p < 1$ and $\frac{1}{2} \leq \delta < 1$.
- Whereas, $\sigma_{B_2C} = \ket{\phi}\bra{\phi}$ with $\ket{\phi} = \sqrt{\alpha}\ket{00} + \sqrt{1-\alpha}\ket{11}$, $\frac{1}{2} \leq \alpha < 1$.
- **Sufficient protocol, $\mathcal{P}_2$:** Bob performs PVM in a complete basis of $\ket{\eta} = \sqrt{\beta}\ket{00} + \sqrt{1-\beta}\ket{11}$ and does CC to Alice. Alice performs a two-outcome POVM and does CC to Charlie. This happens if the parameters p, α, β, δ lie in a specific range,

$$0 < p \leq \frac{1}{3}, \quad \delta_c < \delta < 1, \quad \frac{1}{2} \leq \alpha < \alpha_c(p,\delta), \quad \frac{1}{2} \leq \beta \leq \beta_c(p,\alpha,\delta)$$
$$or \quad \frac{1}{3} < p < 1, \quad \frac{1}{2} \leq \delta < 1, \quad \frac{1}{2} \leq \alpha < \alpha_c(p,\delta), \quad \frac{1}{2} \leq \beta \leq \beta_c(p,\alpha,\delta)$$

- For full expression, refer to Appendix.

Whether there exists a tighter upper bound than $min\{F_1^*,\ F_2^*\}$

- Suppose ρ_{AB_1} and σ_{B_2C} are convex mixtures of Bell basis i.e.,

$$\rho_{AB_1} = \sum_{i=0}^{3} P_i \left|\Phi_i\right\rangle \left\langle\Phi_i\right| , \quad \sigma_{B_2C} = \sum_{j=0}^{3} Q_j \left|\Phi_j\right\rangle \left\langle\Phi_j\right|$$

The best-case fidelity can be achieved as

$$F_{AC}^* \leq \lambda_{max} \left[(\Lambda_{U,\rho,\sigma} \otimes \mathbb{I}) \left|\Phi_0\right\rangle \left\langle\Phi_0\right| \right] \leq \min\{F_1^*,\ F_2^*\},$$

where $\Lambda_{U,\rho,\sigma}$ is a unital qubit channel that depends on ρ_{AB_1} and σ_{B_2C} and $\sqrt{2}\left|\Phi_0\right\rangle = |00\rangle + |11\rangle$. $\lambda_{max}(\rho)$ is the largest eigenvalue of the density matrix ρ.

- ***Sufficient protocol,*** $\mathcal{P}_1$: The protocol that is sufficient to achieve the upper bound is performing complete Bell measurement by Bob followed by local unitary applied by Alice and Charlie.

Bob's measurement in maximally entangled basis is not always optimal

- Previous examples shows Bob's first move while performing PVM in MES is sufficient.
- However, this not always works for any pair of two-qubit states.
- For instance, let $\rho_{AB} = (\Lambda_{ADC} \otimes \mathbb{I}) |\Phi_0\rangle \langle\Phi_0|$ is a Choi state, where Λ_{ADC} is the amplitude-damping channel with channel parameter $\frac{1}{3} < p < 1$.
- Let σ_{B_2C} is two-qubit isotropic state,
  $\sigma_{B_2C} = 0.55 |\Phi_0\rangle \langle\Phi_0| + 0.15 (|\Phi_1\rangle \langle\Phi_1| + |\Phi_2\rangle \langle\Phi_2| + |\Phi_3\rangle \langle\Phi_3|)$.

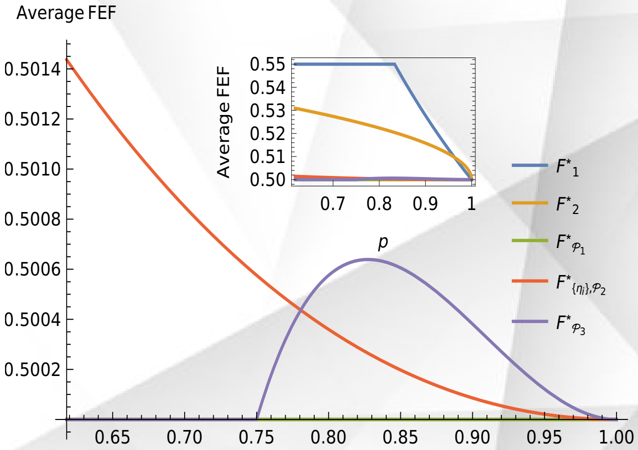
  Efficient protocol: In the above scenario, Bobs first move in doing PVM measurement in non-maximally entangled basis $(Protocol, \mathcal{P}_2)$ is efficient than doing PVM in any MES $(Protocol, \mathcal{P}_1)$. After Bob's measurement, Bob does CC to Alice and Alice does two outcome POVM and does CC to Charlie.

One-way LOCC from Bob to Alice maynot be sufficient

- Let us define another protocol. **Protocol,** $\mathcal{P}_3$: Alice first performs a local filtering operation and classically communicates the outcome to Bob. If the filter passes, all of them applies **Protocol** $\mathcal{P}_1$, otherwise Alice and Charlie discards their existing state and replace it by a pure product state.
- For the same choice of initial states, we see that Bob's first move while performing PVM in any MES basis is not efficient in the entire range $\frac{1}{3} < p < 1$.
- Instead of Bob's first move, if first Alice performs POVM:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1-p}{p^2} \end{pmatrix}, \quad M_1 = \mathbb{I} - M_0, \qquad if \quad \frac{\sqrt{5}-1}{2} < p < 1$$

  and communicates the outcome $\{0, 1\}$ to Bob.
- If the outcome is 0, then Bob performs Bell basis measurement and does CC to Alice and Charlie. Otherwise Bob performs nothing.


Average FEF
0.5014
0.5012
0.5010
0.5008
0.5006
0.5004
0.5002
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
p
Average FEF
0.55
0.54
0.53
0.52
0.51
0.50
0.7
0.8
0.9
1
p
F^*_1
F^*_2
$F^*_{\mathcal{P}_1}$
$F^*_{\{\eta_i\},\mathcal{P}_2}$
$F^*_{\mathcal{P}_3}$

- The small subfigure contains plots of the average *FEF w.r.t.* the channel parameter p of ADC for three different assisted protocols (both PVM and POVM) from Bob to Alice-Charlie.
- It also contains plots of the LOCC upper bounds $F_1^*(\rho, \sigma)$ and $F_2^*(\rho, \sigma)$. The larger figure is a magnified version of the smaller one where the plots of $F_1^*(\rho, \sigma)$ and $F_2^*(\rho, \sigma)$ are omitted.
- Here the **red** curve is for **Protocol** $\mathcal{P}_2$.
- The NME basis for measurement is

$$|\eta_0\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle, \quad |\eta_1\rangle = \frac{1}{2}|00\rangle - \frac{\sqrt{3}}{2}|11\rangle, \quad |\eta_2\rangle = \frac{\sqrt{3}}{2}|01\rangle + \frac{1}{2}|10\rangle, \quad |\eta_3\rangle = \frac{1}{2}|01\rangle - \frac{\sqrt{3}}{2}|10\rangle$$

- This protocol is more efficient than the case when Bob performs PVM in any complete MES basis $\{|\Psi_i\rangle = (U_i \otimes V_i)|\Phi_0\rangle\}$, where $U_i, \ V_i$ are local unitary operations. (**Protocol** $\mathcal{P}_1$).
- The **purple** curve is for **Protocol** $\mathcal{P}_3$.
- The **green** horizontal line is for **Protocol** $\mathcal{P}_1$.
- The plots clearly show that within the region $\frac{\sqrt{5}-1}{2} < p < 1$, the average *FEF* from **Protocol** $\mathcal{P}_3$ can be strictly larger than **Protocol** $\mathcal{P}_1$.

Plan of the talk

- If we consider more than three parties, the situation becomes cumbersome.
- For example, if we consider four parties, just as previously, we can define an optimization problem.
- But this optimization problem will be much more complex than the three parties scenarios. For detail refer to Appendix.
- Thus, a general n-node scenario will be much more non-trivial from the view point of finding an optimal multiparty LOCC protocol.
- We also observe that

Consider a linear network of N parties (nodes), where every segment share a bipartite state $\rho_{i,i+1}$ and all parties can perform LOCC. In such a case the maximum fully entangled fraction that A_1 and A_N can achieve is upper bounded by the LOCC bound

$$F^*_{1N} \leq F^* = \min\{F^*(\rho_{12}),\ F^*(\rho_{23}), ..., F^*(\rho_{N-1N})\}$$

Plan of the talk

n-node scenario

- However there are some instances where n-node scenarios can be handled analytically also.
- We divide the communication line into multiple short-range segments, denoted as S_i, where instead of direct communication from the sender (Alice) to a receiver (Bob), quantum information is relayed through intermediary nodes N_i along these segments.
- Let's assume there are $(n+1)$ segments, where $\{S_1, S_k, S_{n+1}\}_{k=2}^{n}$ are directly controlled by $A - N_1$, $\{N_{k-1} - N_k\}_{k=2}^{n}$, and $N_n - B$ parties, respectively.
- **Question:** If individual segments S_i share some arbitrary state ρ_i, what is the optimal fidelity achievable between the end-to-end parties (AB)?

Upper bound on Optimal Teleportation Fidelity under LOCC

- While general studies are quite nontrivial, for particular two-qubit noises in the given repeater scenario, the upper bound of optimal fidelity can be expressed as follows:

$$F^{\star}_{AB} \leq \min\{F_S, F_C\}.$$

- The bound $F_S = \min_i\{F^{\star}(\rho_i)\}$ arises from $F^{\star}$ being an entanglement measure, where $F^{\star}(\rho_i)$ denotes the OFEF of the state ρ_i. Meanwhile, the $F_C = (1 + \Pi_{i=1}^{n+1}\mathcal{C}_i)/2$ where $\mathcal{C}_i$ is the concurrence of the state ρ_i.
- When a single segment hosts a fixed noisy state ρ_1 with $F^{\star}(\rho_1) \leq F^{\star}(\rho_i)$ for $i \in [2, 4]$, the end-to-end OFEF becomes $F^{\star}_{AB} \leq F^{\star}(\rho)$.
- The upper bound can always be trivially achieved by the E-SWAP protocol utilizing maximal resources (Bell states) in all other segments and Bell measurement at every node but it is not always necessary.

Optimal teleportation fidelity using non-maximal resurces

- ***Proposition***: Consider the free segments $\{S_i\}_{i=2}^{n+1}$, where S_i shares the state $|\psi_i\rangle \equiv \{\sqrt{\alpha_i}, \sqrt{1-\alpha_i}\}$ with $\alpha_i \geq 1/2$, $\forall i$. It is always possible to find noisy states $\rho(p,\delta) = p\mathrm{P} + (1-p)\zeta(\delta)$ shared in the first segment S_1, with non-vanishing region of $\{p,\delta\}$, such that $F^{\star}_{AB} = F^{\star}\left[\rho(p,\delta)\right]$ can be achieved between the sender and receiver.
- ***Sufficient Protocol***: This n-node scenario can be reduced to a three-node scenario through a series of RPBES [9] measurements on nodes $\{N_i\}_{i=2}^{n}$ having two segments S_1 and S_2 with $\rho_{S_1} = \rho(p,\delta)$ and $\rho_{S_2} = \{\sqrt{\alpha'_n}, \sqrt{1-\alpha'_n}\}$ with $\alpha'_n = \dfrac{1+\sqrt{1-\left(\prod_{i=2}^{n+1}\mathcal{C}(\psi_i)\right)^2}}{2}$.
Now as previously we can do two-qubit non-maximally entangled measurements we can establish $F^{\star}_{AB} = F^{\star}(\rho(p,\delta))$.
- To look at full expressions for the bound on state parameters, go to Appendix.

[9] G. Gour and B. C. Sanders, PRL 93, 260501 (2004)

Resource consumption in the proposed LOCC protocol

- If all the segments $\{S_i\}_{i=2}^{n+1}$ share the same non-maximally entangled state, *i.e.*, $\alpha_i = \alpha$ for all $i \in [2, n+1]$ then $\mathcal{C}(\Psi_n) = \mathcal{C}^n$, where $\mathcal{C} = 2\sqrt{\alpha(1-\alpha)}$. We have

$$\frac{1+\sqrt{1-\mathcal{C}^{2n}}}{2} \leq \frac{p^2}{p^2+(1-p)^2\delta(1-\delta)}.$$

- We define a quantity *saved resourcen*:

$$R_v = n(1-\mathcal{C}).$$

- One-*ebit* resource (a maximally entangled state with concurrence $\mathcal{C} = 1$) is required per node in the E-SWAP protocol. Our proposed protocol, which utilizes a non-maximally entangled state with concurrence $\mathcal{C} < 1$, will suffice to satisfy the condition $F^{\star}_{AB} = F^{\star}(\rho(p,\delta))$.
- Thus, R_v quantifies how much less resource is consumed in our proposed protocol compared to the E-SWAP protocol to achieve optimal fidelity in an n-node repeater scenario.

Saved Resource

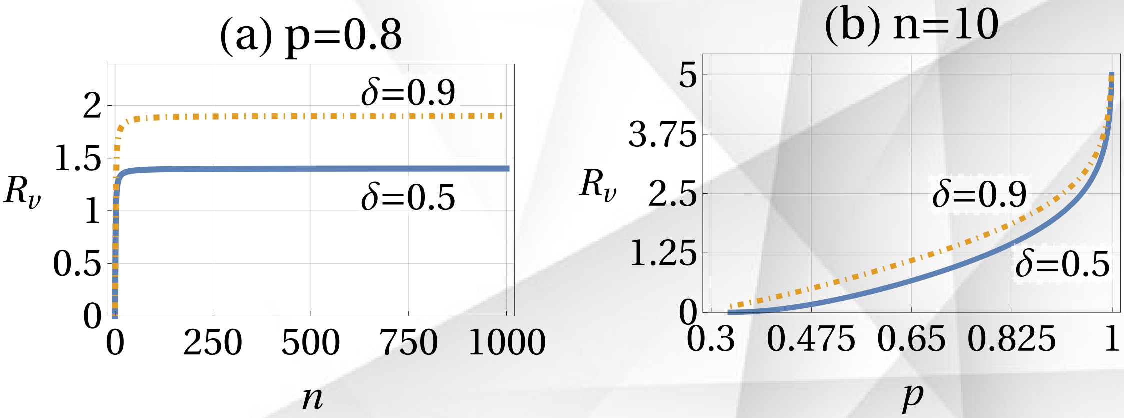


Figure 1: (**a**) Variance in resource consumption with the number of nodes n.(**b**)Variance in resource consumption with the noise parameter p. For each case $\beta = \frac{1}{2}$.

Effect of the position of noisy node on Resource Consumption

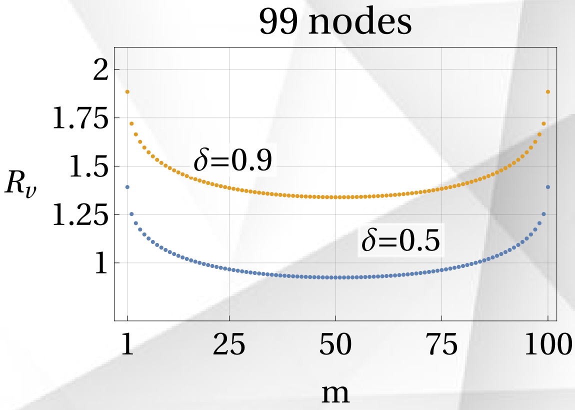


Figure 2: Variance in resource consumption with the position of the noisy node m for the given number of nodes $n = 99$ and the noise parameter $p = 0.8$. This figure is generated for two different values of Schmidt coefficient δ of the entangled state in $\rho(p, \delta)$. For each case $\beta = \frac{1}{2}$.

- To get a more intuitive picture of the exact meaning of the quantity R_v we first consider a three-node scenario where state in noisy segment is $\rho_{AN_1} = \rho(p,\delta)$ and state $\rho_{N_1B} = \chi$ in free segment is noisy but entangled instead of pure.
- Multiple copies of this entangled resource χ can be used to improve the communication channel.
- A traditional approach involves distilling χ into maximally entangled states $|\phi^+\rangle$.
- We consider $\chi = F\,|\Psi^-\rangle\langle\Psi^-| + \frac{(1-F)}{3}\Big(|\Psi^+\rangle\langle\Psi^+| + |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|\Big)$ as a Werner state and use the hashing protocol[10].
- Remember our protocol suggests that a non-maximally entangled state $|\psi\rangle = \sqrt{\alpha}\,|00\rangle + \sqrt{1-\alpha}\,|11\rangle$ can sufficiently establish the same average teleportation fidelity as that of a maximally entangled state in E-SWAP.
- Thus, achieving optimal teleportation fidelity may be more efficient by distilling a specific non-maximally entangled state $|\psi\rangle$ instead of a maximally entangled state:

$$(\chi)^{\otimes j} \to \left(|\phi^+\rangle\langle\phi^+|\right)^{\otimes k} \leftrightarrow \left(|\psi\rangle\langle\psi|\right)^{\otimes k'}.$$

[10]. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K., Phys. Rev. A 54, 3824 (1996)

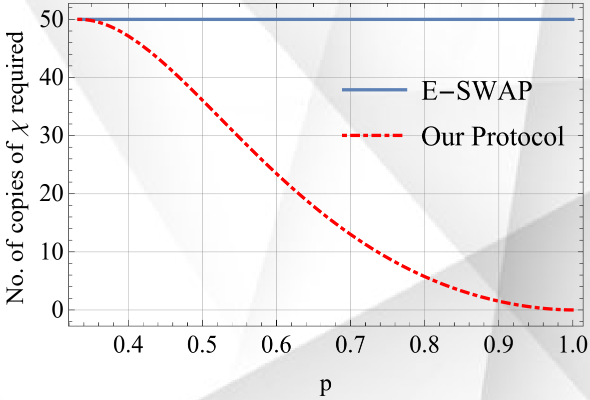


Figure 3: Noise vs. Copies : This figure illustrates how the number of Werner states required as resources in the free segment decreases as the noise in the noisy segment increases for fixed value of $F = 0.8161$. Unlike in traditional entanglement swapping protocols, where the number of required copies remains independent of noise, our protocol shows that higher noise in the noisy segment allows for a reduction in the number of copies needed in the free segment to achieve the same level of teleportation fidelity.

- From the above figure, we can intuitively understand how noise in the free segments reduces the number of required copies.
- It is noteworthy that the conventional E-SWAP protocol does not have this flexibility and remains independent of noise.
- Detailed calculations can be found in Appendix.
- Furthermore, suppose we have n free segments in a linear network with all free segments sharing the same non-maximally entangled state with a Schmidt coefficient α.
- If we aim to distill this state in each free segment from an entangled resource χ, the number of copies of χ required in the n node setup in term of saved resource R_v is given by Fig. (4)

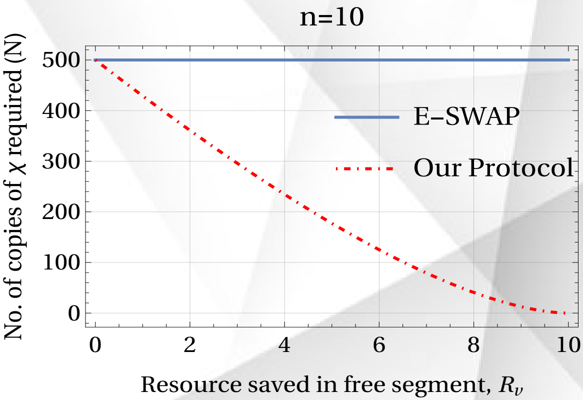


Figure 4: Resource Saved vs. Number of Copies : This figure illustrates how the number of Werner states required as resources in the free segment decreases due to noise. Consequently, the amount of resource saved, R_v, in terms of entanglement, increases in the free segment. The total number of segments considered here is 10. This plot provides a more clear understanding of the nature of the resource savings achieved.

Meaning of the saved resource R_v

- This Figure is also plotted for a fixed fidelity value $F = 0.8161$.
- It clearly illustrates the inverse proportionality between the number of required copies and the saved resource in our protocol.
- As we understand, the number of required copies per segment decreases due to noise, leading to a reduction in the total resource required compared to the ESWAP protocol.
- Consequently, an amount of resource equal to $(1 - C(\alpha))$ is saved at each free segment, and the total saved resource over n free segments is given by

$$R_v = n(1 - C(\alpha)).$$

Noise analysis

- We also considered the scenarios where the state in the free segment is noisy but only a single copy of it is available.
- We considered two types of noises:
  - Photon-Loss Noise

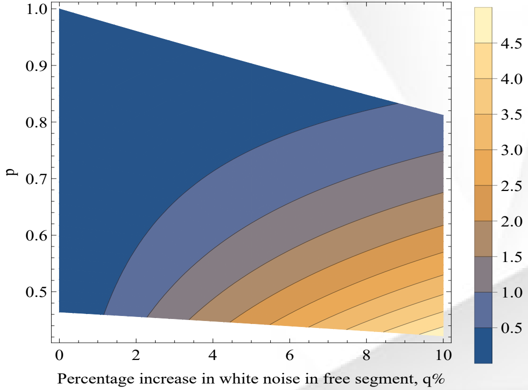
$$\chi = (1-q) \left|\Psi^+\right\rangle \left\langle\Psi^+\right| + q \left|00\right\rangle \left\langle 00\right| .$$

  - White noise

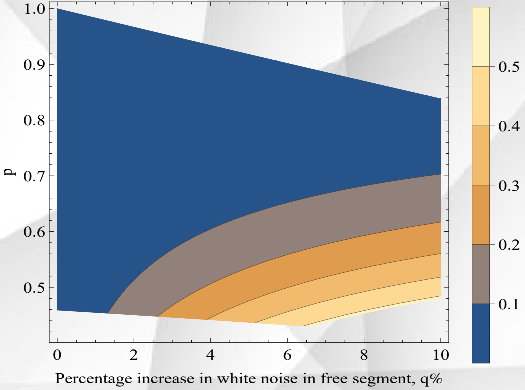
$$\chi = (1-q) \left|\Phi^+\right\rangle \left\langle\Phi^+\right| + q\frac{\mathbb{I}}{4}.$$

- To quantify the effect of noise, we focus on the percentage change in fidelity.
- We plot this quantity with both the percentage increase in the noise injected in the free segment and the noise parameter of the state in the noisy segment.

White Noise



(a)

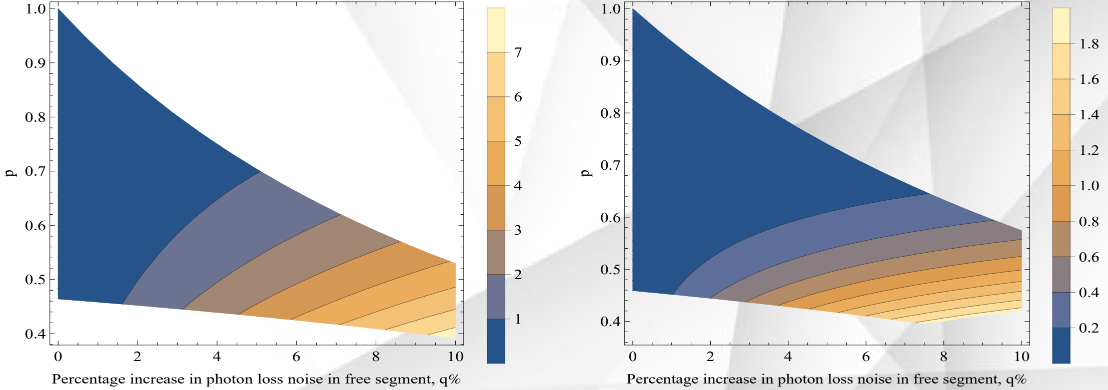


(b)

Figure 5

- **(a)** Percentage change in fidelity, $F\% = \frac{F^*_{AB} - F^{\text{noisy}}_{AB}}{F^*_{AB}} \times 100$, as a function of the noise parameter p in the shared state within the noisy segment and the noise parameter q induced in the free segment. Notably, as p increases, the percentage change in fidelity decreases. Additionally, even for sufficiently high values of q, the change in fidelity remains relatively low.
- Percentage change in fidelity, $F\% = \frac{F^{\text{ME}}_{AB} - F^{\text{NME}}_{AB}}{F^{\text{ME}}_{AB}} \times 100$, when white noise (q) is added to the maximally entangled state compared to the non-maximally entangled state in the free segment, along with an increase in the noise parameter (p) of the state in the noisy segment. Similar phenomena are observed in the earlier plot. Both plots suggest that the noise in the noisy segment and the noise injected in the free segment are not arbitrary but are interrelated. Depending on the permissible tolerance for the fidelity change, a trade-off exists between these two noise parameters.

Photon-Loss Noise



(a) (b)

Figure 6

- **(a)** Percentage change in fidelity $F\% = \frac{F^{*}_{AB} - F^{noisy}_{AB}}{F^{*}_{AB}} \times 100$ with an increase in the noise parameter p of the state in the noisy segment and percentage increase of photon loss noise in the free segment.
- **(b)** Percentage change in fidelity $F\% = \frac{F^{ME}_{AB} - F^{NME}_{AB}}{F^{ME}_{AB}} \times 100$ when photon loss noise is added to the maximally entangled state as compared to the case when the same amount of photon loss noise is added to the non-maximally entangled state in the free segment along with an increase in the noise parameter p of the state in the noisy segment. Again, both of these plots indicate that the amount of noise in the noisy segment and the amount of noise present in the free segment depend on each other. Depending on the amount of change in the value of fidelity defined as permissible, there exists a trade-off between these two. Also comparing it with the previous Figure we observe that fidelity is more robust under white noise injection as compared to the photon loss noise.

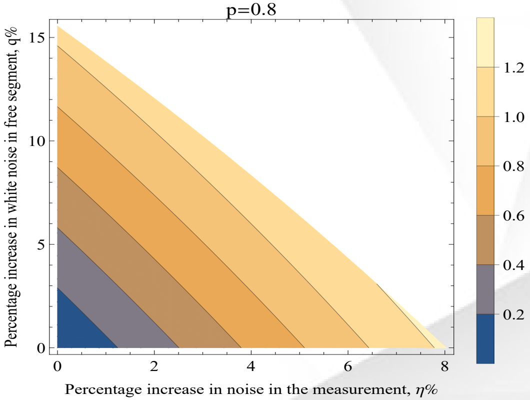
- We can consider a scenario where instead of a projective measurement in the computational basis we can have a POVM given as

$$|\Pi_0\rangle\langle\Pi_0| = (1-\eta)|0\rangle\langle 0| + \eta|1\rangle\langle 1| \quad (1)$$

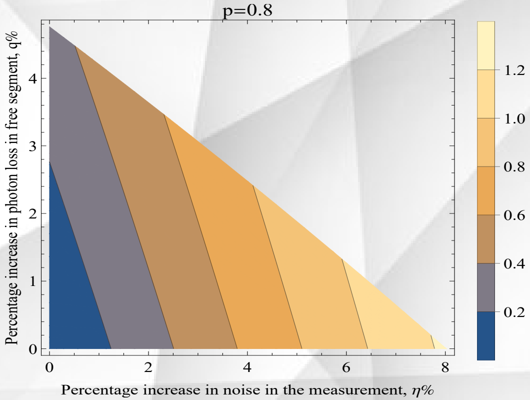
$$|\Pi_1\rangle\langle\Pi_1| = (1-\eta)|1\rangle\langle 1| + \eta|0\rangle\langle 0| \quad (2)$$

- The free segment can still have either *Photon-Loss noise* or *White Noise*.
- However for plotting purposes the noise in the noisy segment has been fixed to a certain value.

Noisy Measurement



(a)



(b)

Figure 7

- **(a)** Percentage change in fidelity $F\% = \frac{F^*_{AB} - F^{noisy}_{AC}}{F^*_{AB}} \times 100$ with an increase in the percentage of noise in the measurement and percentage of white noise added in the free segment. The noise parameter of the state in the noisy segment p is taken to be 0.8.
- **(b)** Percentage change in fidelity $F\% = \frac{F^*_{AB} - F^{noisy}_{AC}}{F^*_{AB}} \times 100$ with an increase in the percentage of noise in the measurement and percentage of colored noise added in the free segment. Again, the noise parameter p of the state in the noisy segment is 0.8. From both of these plots, we can infer that the noise in the measurement and the noise in the free segment constitute an interplay between them for a given amount of tolerance on the change in fidelity from the optimal value. This feature will be present for any allowed value of the noise parameter p in the noisy segment.

Plan of the talk

Summary

- We study entanglement distribution over a linear network. This is because entanglement is a necessary resource for quantum coummunication.
- There are no unique measures of entanglement for mixed states. Due to its operational nature, we focused on maximizing teleportation fidelity between two end nodes.
- What we found is that the distribution of optimal teleportation channel is not trivial.
- However, we found that in a three-node scenario, it is sufficient that at least one of the extreme node performs a trivial operation.
- For a three-node scenario, we propose some possible upper bounds on the optimal teleportation fidelity (OTF), which are independent of each other.
- We explore the complexity beyond the three-node scenario.

Summary

- We then studied a n-node repeater scenario.
- We show that with less entanglement in free segments, we can establish the same teleportation fidelity as that could have been established with maximally entangled states.
- We then analyzed how this saved resource with parameters of states, number of nodes, and position of the noise in the linear repeater network.
- We also did a noise analysis which tracks the change in fidelity in the presence of noise in the free segments instead of pure states.

Future Scope

- We can explore this problem for different network structures.
- We can study the distribution of teleportation channels while dealing with infinite-dimensional quantum systems.
- We can look for other operational measures of entanglement which can be distributed across the network with fewer resources.

- THE END -

Thank you for your attention

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APPENDIX

Standard Teleportation protocol

- An arbitrary two-qubit state ρ has a general form

$$\rho_{AB} = \frac{1}{4}\left[\mathbb{I}\otimes\mathbb{I} + \mathbf{R}.\sigma\otimes\mathbb{I} + \mathbb{I}\otimes\mathbf{S}.\sigma + \sum_{i,j=1}^{3} T_{ij}\sigma_i\otimes\sigma_j\right]$$

- ***Quantum advantage*** : Under local unitary manipulation, ρ_{AB} is **useful** for QT *iff* $\mathcal{F}_{\rho_{AB}} > \frac{2}{3}$
- ***Standard protocol***:

$$\rho_{AB} \longrightarrow \tilde{\rho}_{AB} \quad (local\ unitary\ transformation)$$

$$where \quad \tilde{\rho}_{AB} = \frac{1}{4}\left[\mathbb{I}\otimes\mathbb{I} + \mathbf{r}.\sigma\otimes\mathbb{I} + \mathbb{I}\otimes\mathbf{s}.\sigma + \sum_{i=1}^{3} t_{ii}\sigma_i\otimes\sigma_i\right]$$

$$such\ that \quad \mathcal{F}_{\rho_{AB}} = \langle f_{\tilde{\rho}_{AB}}\rangle$$

Maximal Fidelity value

- For any of transformed state $\rho_{AB} \longrightarrow \tilde{\rho}_{AB}$, the maximal fidelity value can be computed as

$$\mathcal{F}_{\rho_{AB}} = \frac{1}{2}\left[1 + \frac{\sum_{i=1}^{3}|t_{ii}|}{3}\right] \qquad if \quad t_{11}t_{22}t_{33} \leq 0$$
$$= \frac{1}{2}\left[1 + \frac{\max_{i\neq j\neq k=1}(|t_{ii}| + |t_{jj}| - |t_{kk}|)}{3}\right] \quad if \quad t_{11}t_{22}t_{33} > 0$$

- Any ρ_{AB} with $det[T(\rho_{AB})] < 0$ is always useful for QT under standard protocol.
- However,converse of this statement is not always true in general.
- The necessary and sufficient condition for usefulness of ρ_{AB} under standard protocol is $\sum_{i=1}^{3}|t_{ii}| > 1$.

Canonical form of two-qubit states under SLOCC

- The operation $(A \otimes \mathbb{I})$ transforms Hilbert-Schmidt basis of ρ_{AB} as [11]

$$
\begin{aligned}
(A \otimes \mathbb{I})\ \rho_{AB}\ (A^{\dagger} \otimes \mathbb{I}) &= \sum_{\mu,\nu=0}^{3} \mathcal{R}_{\mu,\nu}\ A\sigma_{\mu}A^{\dagger} \otimes \sigma_{\nu}, \quad \text{where} \quad \mathcal{R}_{\mu\nu} = Tr\left[\rho_{AB}\ (\sigma_{\mu} \otimes \sigma_{\nu})\right] \\
&= \sum_{\mu,\nu=0}^{3} \mathcal{R}_{\mu\nu} \sum_{\alpha=0}^{3} L_{\alpha\mu}\ \sigma_{\alpha} \otimes \sigma_{\nu} \quad \text{where} \quad L_{\alpha\mu} \in \mathbb{R},
\end{aligned}
$$

- Hence, if $\mathcal{R}(\rho_{AB}) = (\mathcal{R}_{\mu\nu})$ represents ρ_{AB}, then $(A \otimes \mathbb{I})$ transforms $\mathcal{R}$ as $\tilde{\mathcal{R}}(\sigma_{AB}) = L_A\ \mathcal{R}(\rho_{AB}) L_A^T$, where $L_A \in$ orthochronous proper Lorentz group (OPLG) that is $SO(3,1)$.

- **OPLG**: A set of 4×4 real matrix $\{L \mid det(L) = 1,\ L_{00} > 0,\ L^T\ G\ L = G = diag\{1, -1, -1, -1\}\}$

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[11] F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003), F. Verstraete, J. Dehaene, and B. DeMoor, Phys. Rev. A 64, 010101(R) (2001).

Canonical form of two-qubit states under SLOCC

- ***Question***: Given a two-qubit state ρ_{AB}, how to obtain its canonical representation?

  Answer: If $\mathcal{R}(\rho)$ is the representation of ρ_{AB}, then evaluate the eigenvalues and eigenvectors of the matrix: $M_\rho = G\,\mathcal{R}(\rho)\,G\,\mathcal{R}(\rho)^T$ [12]

- Suppose M_ρ has four non-degenerate eigenvalues, $\lambda_0 > \lambda_1 > \lambda_2 > \lambda_3$.

- If $L_A \in OPLG$ transforms $\mathcal{R}(\rho)$ to its canonical form then

$$M_\rho\, a_\mu = \lambda_\mu\, a_\mu, \quad s.t. \quad a_\mu^T\, G\, a_\nu = G_{\mu\nu} \quad and \quad L_A^T \equiv (a_0 \;\; a_1 \;\; a_2 \;\; a_3)$$

- One of the canonical representation (full rank) of ρ_{AB}:

$$\rho_{AB}^c = \frac{(A \otimes \mathbb{I})\;\rho\;(A^\dagger \otimes \mathbb{I})}{p} = \frac{1}{4}\left[\mathbb{I}_4 + \sum_{i=1}^{2}\sqrt{\frac{\lambda_i}{\lambda_0}}\,\sigma_i \otimes \sigma_i - \sqrt{\frac{\lambda_3}{\lambda_0}}\,\sigma_3 \otimes \sigma_3\right]$$

---

[12] Sudha et al., Phys. Rev. A 102, 052419 (2020).

How to implement optimal LOCC ?

- In general it is difficult to answer what is optimal implementation.
- Even a finite $n-$round LOCC on a bipartite system is a strict subset of LOCC i.e., $LOCC_N \subset LOCC$.
- Any LOCC can be expressed as a Separable operation (SEP), converse is not true in general because $LOCC \subset SEP$.
- SEP has a simple operator structure and therefore one can say

$$F^*_{SEP}(\rho_{AB_1}, \sigma_{B_2C}) = \max_{|\Phi\rangle \in MES} \max_{\Lambda \in SEP} \langle \Phi_{AC} | Tr_B \left(\Lambda(\eta_{AB_1B_2C}) \right) | \Phi_{AC} \rangle$$

$$= \max_{|\Phi\rangle} \max_{\{m_i, n_i, o_i\}} \langle \Phi_{AC} | Tr_B \left(\sum_i (m_i \otimes n_i \otimes o_i) \, \eta_{AB_1B_2C} \, (m_i \otimes n_i \otimes o_i)^\dagger \right) | \Phi_{AC} \rangle$$

$$= \max_{U_A,\, V_C} \max_{\{m_i,\, n_i,\, o_i\}} \langle \Phi_0 | (U_A^\dagger \otimes V_C^\dagger) \left(\sum_i (m_i \otimes o_i) \, Tr_B \left((\mathbb{I} \otimes n_i^\dagger n_i \otimes \mathbb{I}) \, \eta_{AB_1B_2C} \right) (m_i \otimes o_i)^\dagger \right) (U_A \otimes V_C) | \Phi_0 \rangle$$

$$
\begin{aligned}
&= \max_{\{m_i,\ n_i,\ o_i\}} \sum_i \langle \Phi_0 | \left(m_i \otimes o_i\right) Tr_B \left(\left(\mathbb{I} \otimes n_i^\dagger n_i \otimes \mathbb{I}\right) \eta_{AB_1B_2C} \right) \left(m_i \otimes o_i\right)^\dagger) | \Phi_0 \rangle \\
&= \max_{\{m_i,\ N_i,\ o_i\}} \sum_i \ \langle \Phi_0 | \left(o_i^T \ m_i \otimes \mathbb{I}\right) \tilde{\chi}_{AC}(N_i) \ \left(m_i^\dagger \ o_i^* \otimes \mathbb{I}\right) | \Phi_0 \rangle \,, \qquad \text{where} \quad N_i = n_i^\dagger n_i, \\
&= \max_{\{m_i,\ N_i\}} \ \langle \Phi_0 | \left(\sum_i \ \left(m_i \otimes \mathbb{I}\right) \tilde{\chi}_{AC}(N_i) \ \left(m_i^\dagger \otimes \mathbb{I}\right) \right) | \Phi_0 \rangle \,, \\
&= \max_{\{m_i,\ N_i\}} \ Tr \left(\sum_i \ \left(m_i \otimes \mathbb{I}\right) \tilde{\chi}_{AC}(N_i) \ \left(m_i^\dagger \otimes \mathbb{I}\right) P_{\Phi_0} \right) \\
&= \max_{\{m_i,\ N_i\}} \sum_i \ Tr \left(\tilde{\chi}_{AC}(N_i) \ X_{m_i} \right) \geq F^*_{LOCC}(\rho_{AB_1}, \sigma_{B_2C}).
\end{aligned}
$$

Click here to go back.

The optimization problem under SEP

- For any given states ρ_{AB_1} and σ_{B_2C}, one can propose an optimization problem under three-party SEP operation to obtain $F^*_{SEP}(\rho_{AB_1}, \sigma_{B_2C})$ as follows:

$$\max_{\{N_i, m_i\}} \quad \sum_{i=1}^{K} \langle\ \tilde{\chi}_{AC}(N_i), X_{m_i}\ \rangle$$

$$s.t. \quad \sum_{i=1}^{K} M_i = m_i^\dagger m_i = \mathbb{I}_d, \quad \sum_{i=1}^{K} N_i = n_i^\dagger n_i = \mathbb{I}_{d^2}$$

$$N_i \geq 0, \quad M_i \geq 0 \qquad \forall i$$

$$\sum_{i=1}^{K} \tilde{\chi}_{AC}(N_i) = Tr_B(\rho_{AB_1} \otimes \sigma_{B_2C}), \quad \tilde{\chi}_{AC}(N_i) \geq 0 \quad \forall N_i$$

$$X_{m_i} = (m_i^\dagger \otimes \mathbb{I}_d) \left|\Phi_0\right\rangle \left\langle\Phi_0\right| (m_i \otimes \mathbb{I}_d) \geq 0 \quad \forall i$$

- The given problem has two independent variables N_i and M_i.
- Now let us first choose any set of $\{m_i\}$ which fixes the set of unnormalized projectors $\{X_{m_i}\}$ and for simplicity, let us denoted every $X_{m_i} = X_i$.
- We now have to solve a convex optimization problem over the set of variables $\{\tilde{\chi}_{AC}(N_i)\}$. For simplicity, let us denote $\{\tilde{\chi}_{AC}(N_i) = \tilde{\chi}_i\}$ as a convex set of K POVM elements. Now define

$$X = \bigoplus_{i=1}^{K} X_i, \quad Y = \bigoplus_{i=1}^{K} \tilde{\chi}_i$$

and let Φ be a linear mapping defined as $\Phi(Y) = \sum_i \tilde{\chi}_i = Tr_B(\rho_{AB_1} \otimes \sigma_{B_2C})$. Now the following convex optimization problem can be proposed as

$$\begin{aligned} (i) \quad & G_X(\rho_{AB_1}, \sigma_{B_2C}) = \max_Y Tr\,[X\ Y] \\ s.t. \quad & \Phi(Y) = \sum_{i=1}^{K} \tilde{\chi}_i = \eta_{AC} = Tr_B(\rho_{AB_1} \otimes \sigma_{B_2C}) \\ & and \quad Y \geq 0 \end{aligned}$$

- After solving this convex optimization problem one can come up with a set of optimal feasible solution, say $Y^*(X) = \{\tilde{\chi^*}_i(X_i)\}$ that explicitly depends on the set $X = \{X_i\}$.
- Now to evaluate $F^*_{SEP}(\rho_{AB_1}, \sigma_{B_2C})$ one needs to solve another optimization problem,

$$(ii) \quad F^*_{SEP}(\rho_{AB_1}, \sigma_{B_2C}) = \max_{X=\bigoplus X_i} \quad G_X(\rho_{AB_1}, \sigma_{B_2C})$$

$$s.t. \quad \mathbb{I}_{d^2} \geq X_i \geq 0, \quad rank(X_i) = 1 \qquad \forall i$$

- Notice that the problem (i) is a convex optimization problem (ii) is not as the individual X_i is only restricted to rank one POVM. Hence, the constraint of (ii) is not convex.

Dual problem

- As per (i), one can construct the Lagrangian as

$$\mathcal{L}_X = \sum_{i=1}^{K} Tr\left[X_i \ \tilde{\chi}_i\right] + Tr\left[\Xi \left(\eta_{AC} - \sum_{i=1}^{K} \tilde{\chi}_i\right)\right] + \sum_{i=1}^{K} Tr\left[\ \Xi_i \ \tilde{\chi}_i\right], \quad (3)$$

  where Ξ and $\{\Xi_i\}$ are positive Hermitian operators and known as the Lagrange multipliers.
- The dual objective function $H_X(\rho_{AB_1}, \sigma_{B_2C})$ of the primal objective $G_X(\rho_{AB_1}, \sigma_{B_2C})$ can be expressed as

$$H_X(\rho_{AB_1}, \sigma_{B_2C}) = \max_{Y=\{\tilde{\chi}_i\}} \mathcal{L}_X \geq G_X(\rho_{AB_1}, \sigma_{B_2C})$$

- The dual optimization problem can be proposed as

$$(iii) \quad H_X(\rho_{AB_1}, \sigma_{B_2C}) = \min_{\Xi} \quad Tr\,[\Xi\; \eta_{AC}]$$
$$s.t. \quad X_i \;\leq \Xi \quad and \quad \Xi \geq 0 \qquad \forall i$$

- As already discussed (i) is convex and the objective function is linear in Y. So without any loss of generality if we choose

$$Y = \bigoplus_{i=1}^{K} \frac{\eta_{AC}}{K}, \quad \Xi = M\,\mathbb{I},$$

where $M \geq 1$ then we can always conclude that Slater's condition [13] is satisfied irrespective of the choice of X which renders

$$H_X(\rho_{AB_1}, \sigma_{B_2C}) = G_X(\rho_{AB_1}, \sigma_{B_2C})$$

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[13] S. Boyd and L. Vandenberghe, Convex optimization (Cam- bridge university press, 2004).

Conditions for achieving optimal teleportation fidelity

- The conditions on all these state parameters for achieving optimal teleportation fidelity are given as

$$0 < p \leq \frac{1}{3}, \quad \frac{1}{2}\left(1 + \sqrt{1 - \frac{4p^2}{(1-p)^2}}\right) < \delta < 1, \quad \frac{1}{2} \leq \alpha < \frac{p^2}{p^2 + (1-p)^2\delta(1-\delta)},$$
$$\frac{1}{2} \leq \beta \leq \frac{p^2(1-\alpha)}{p^2(1-\alpha) + (1-p)^2\alpha\delta(1-\delta)},$$

or,

$$\frac{1}{3} < p < 1, \quad \frac{1}{2} \leq \delta < 1, \quad \frac{1}{2} \leq \alpha < \frac{p^2}{p^2 + (1-p)^2\delta(1-\delta)},$$
$$\frac{1}{2} \leq \beta \leq \frac{p^2(1-\alpha)}{p^2(1-\alpha) + (1-p)^2\alpha\delta(1-\delta)}.$$

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Four party scenario

- Let us consider a four-party network i.e., $N = 4$ and they initially share a four party state

$$\rho_{1234} = \rho_{12} \otimes \rho_{23} \otimes \rho_{34}$$

- Suppose, all the parties perform an operation Λ which prepares a bipartite state between A_1 and A_4.
- Fully entangled fraction under Λ operation is

$$F_{\Lambda,14} = \max_{|\Phi\rangle} \langle\Phi| \Lambda(\rho_{1234}) |\Phi\rangle$$

$$= \max_{|\Phi\rangle} \sum_{m=1}^{K} \langle\Phi| \; Tr_{23} \left[\left(\bigotimes_{i=1}^{4} K_i^m \right) \rho_{1234} \left(\bigotimes_{i=1}^{4} K_i^m \right)^\dagger \right] |\Phi\rangle$$

$$= \max_{|\Phi\rangle} \sum_{m=1}^{K} \langle\Phi| (K_1^m \otimes K_4^m) \, \chi_{14}(M_2^m, M_3^m)(K_1^m \otimes K_4^m)^\dagger |\Phi\rangle ,$$

- $M_{2,3}^m = (K_{2,3}^m)^\dagger \; K_{2,3}^m$ is the $K-$outcome two-qudit POVM acting on A_2 and A_3 and also note that $\chi_{14}(M_2^m, M_3^m)$ is a bipartite state that can be written as

$$\chi_{14}(M_2^m, M_3^m) = Tr_{23}\left[(\mathbb{I}_{14} \otimes M_2^m \otimes M_3^m) \; \rho_{1234}\right],$$

where $M_{2,3} \in \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d)$.

- Thus one can conclude that the optimal fully entangled fraction under SEP operation, $F^*_{SEP,14}$ can be expressed as

$$\max_{\{K_1^m, M_2^m, M_3^m\}} \sum_{m=1}^{K} \langle \Phi_0 | \, (K_1^m \otimes \mathbb{I}_4) \chi_{14}(M_2, M_3) (K_1^m \otimes \mathbb{I}_4)^\dagger \, | \Phi_0 \rangle$$
$$= \max_{\{K_1^m, \; M_2^m, \; M_3^m\}} \sum_{m=1}^{K} Tr\left[\chi_{14}(M_2^m, M_3^m) \; X(K_1^m)\right],$$

- Now let us define two operators,

$$Y = \bigoplus_{m=1}^{K} \chi_{14}(M_1^m, M_2^m), \quad X = \bigoplus_{m=1}^{K} X(K_1^m).$$

- Similar as $N = 3$, one can propose an optimization problem for $N = 4$ as

$$\begin{aligned}
\max \quad & Tr\ [YX] \\
s.t. \quad & \sum_{m=1}^{K} M_2^m = \sum_{m=1}^{K} M_1^m = \mathbb{I}_{d\times d} \\
& \sum_{n=1}^{K} \chi_{14}(M_2^n, M_3^m) = Tr_{23}\left[(\mathbb{I}_{14} \otimes \mathbb{I}_2 \otimes M_3^m)\ \rho_{1234}\right] \\
& \sum_{p=1}^{K} \chi_{14}(M_2^m, M_3^p) = Tr_{23}\left[(\mathbb{I}_{14} \otimes M_2^m \otimes \mathbb{I}_3)\ \rho_{1234}\right] \\
& \chi_{14}(M_2^m, M_3^m) \geq 0 \qquad \forall m \\
& X(K_1^m) \geq 0, \quad rank[X(K_1^m)] = 1 \quad \forall m
\end{aligned}$$

- Notice from the above optimization problem that the objective function is not linear in terms of both the variables M_2^m and M_3^m.
- However, for given choice of M_2^m the objective function becomes linear with respect to M_3^m and vice versa.
- The last constraint is not convex, hence the optimization is not a convex optimization and it has an increasing complexity in comparison with $N = 3$.
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Bounds on the state parameters for achieving OTF

- For simplicity we implement Bell measurement($\beta = \frac{1}{2}$). The bounds on the state parameters are given as

$$0 < p \leq \frac{1}{3}, \qquad \frac{1}{2}\left(1 + \sqrt{1 - \frac{4p^2}{(1-p)^2}}\right) < \delta < 1, \qquad \frac{1}{2} \leq \alpha_n' \leq \frac{p^2}{p^2 + (1-p)^2\delta(1-\delta)},$$

or,

$$\frac{1}{3} < p < 1, \qquad \frac{1}{2} \leq \delta < 1, \qquad \frac{1}{2} \leq \alpha_n' \leq \frac{p^2}{p^2 + (1-p)^2\delta(1-\delta)}.$$

- From the third inequality, we have

$$\frac{1 + \sqrt{1 - \left(\prod_{i=2}^{n+1} \mathcal{C}(\psi_i)\right)^2}}{2} \leq \frac{p^2}{p^2 + (1-p)^2\delta(1-\delta)}.$$

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- To get a more intuitive picture of the exact meaning of the quantity R_v we consider the scenarios where free segments have noisy states χ shared instead of pure non-maximally entangled states.
- In the free segment, multiple copies of the entangled resource χ can be used to improve the communication channel.
- A traditional approach involves distilling χ into maximally entangled states $|\phi^+\rangle$.
- However, achieving optimal teleportation fidelity may be more efficient by distilling a specific non-maximally entangled state $|\psi\rangle$ instead of a maximally entangled state:

$$(\chi)^{\otimes j} \rightarrow \left(|\phi^+\rangle\langle\phi^+|\right)^{\otimes k} \leftrightarrow \left(|\psi\rangle\langle\psi|\right)^{\otimes k'} .$$

- Here, the first transformation, which distills maximally entangled states from χ, is not necessarily reversible for general states χ, even in the asymptotic limit [14].
- On the other hand, the second transformation, converting maximally entangled states to non-maximally entangled states, is reversible in the asymptotic limit [15].

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[14]. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009)

[15] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996)

- For our analysis, we consider $\chi = F\left|\Psi^-\right\rangle\left\langle\Psi^-\right| + \frac{(1-F)}{3}\Big(\left|\Psi^+\right\rangle\left\langle\Psi^+\right| + \left|\Phi^+\right\rangle\left\langle\Phi^+\right| + \left|\Phi^-\right\rangle\left\langle\Phi^-\right|\Big)$ as a Werner state
- Distillation rate to get maximally entangled state Φ^+ from it is given as $D_{\Phi^+}(\chi) = 1 + F\log_2(F) + (1-F)\log_2\left(\frac{1-F}{3}\right)$ using the hashing protocol[16].
- If we follow the traditional protocol to establish optimal teleportation by distilling maximally entangled states in the free segment then the number of copies of χ required in the process is given by $\frac{1}{D_{\phi^+}(\chi)}$, which does not depend on the noisy state $\rho(p,\delta) = p\left|01\right\rangle\left\langle 01\right| + (1-p)\left|\zeta(\delta)\right\rangle\left\langle\zeta(\delta)\right|$ in the noisy segment.
- However, our protocol suggests that a non-maximally entangled state $\left|\psi\right\rangle = \sqrt{\alpha}\left|00\right\rangle + \sqrt{1-\alpha}\left|11\right\rangle$ can sufficiently establish the same average teleportation fidelity with the noisy state $\rho(p,\delta)$. The distillation rate of a non-maximally entangled state is given by:

$$D_\psi(\chi) = \frac{D_{\phi^+}(\chi)}{S(\rho_\psi)}.$$

16. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K., Phys. Rev. A 54, 3824 (1996)

- For a non-maximally entangled state $|\psi\rangle$, the von Neumann entropy $S(\rho_\psi) < 1$, which implies that $D_\psi(\chi) > D_{\phi^+}(\chi)$.
- Therefore, in the asymptotic limit, preparing one copy of $|\psi\rangle$ requires $\frac{1}{D_\psi(\chi)}$ copies of χ, which is fewer than the $\frac{1}{D_{\phi^+}(\chi)}$ copies needed to prepare a maximally entangled state.
- More precisely

$$\frac{1}{D_\psi(\chi)} = \frac{S(\rho_\psi)}{D_{\phi^+}(\chi)} = \frac{4p^2 \log_2\left(\frac{4p^2}{5p^2-2p+1}\right) + (p-1)^2 \log_2\left(\frac{(p-1)^2}{5p^2-2p+1}\right)}{(5p^2-2p+1)\left(-F\log_2(1-F) + \log_2\left(-\frac{2}{3}(F-1)\right) + F\log_2(3F)\right)}.$$

- Furthermore, suppose we have n free segments in a linear network with all free segments sharing the same non-maximally entangled state with a Schmidt coefficient α.
- If we aim to distill this state in each free segment from an entangled resource χ, which in this section is taken to be a Werner state, the number of copies of χ required in the n node setup in term of saved resource R_v is given by
-

$$N = \frac{n}{D_\psi(\chi)} = \frac{nS(\rho_\psi)}{D_{\phi^+}(\chi)} = \frac{n\left(-\left(\frac{1+\sqrt{1-(1-\frac{R_v}{n})^2}}{2}\right)\log_2\left(\frac{1+\sqrt{1-(1-\frac{R_v}{n})^2}}{2}\right) - \left(\frac{1-\sqrt{1-(1-\frac{R_v}{n})^2}}{2}\right)\log_2\left(\frac{1-\sqrt{1-(1-\frac{R_v}{n})^2}}{2}\right)\right)}{\left(1 + F\log_2(F) + (1-F)\log_2\left(\frac{1-F}{3}\right)\right)}$$

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