Basics of Lattice-Based Cryptography

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What is Cryptology?

Cryptology is the science of secrecy.

Paradigms of Cryptography

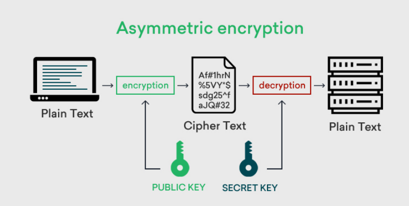

Figure: Public-key cryptography (RSA, El-Gamal, Diffie-Hellman)

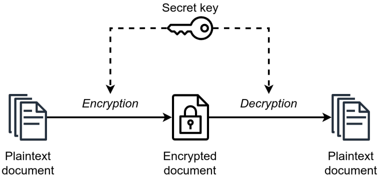

Figure: Symmetric-key cryptography (AES, DES, CBC, OCB)

Computational Assumption on the Security of Cryptographic Scheme

- Security of almost all cryptographic schemes are based on mathematical problems that are computationally difficult for classical computers to solve.
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Quantum computers can solve some of this mathematical hard problem efficiently

Quantum Algorithms

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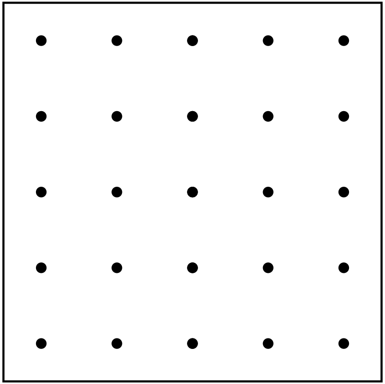
We require cryptographic schemes which are secure in the presence of quantum algorithms.

Lattice Based Cryptography is one of the possible candidates which are believed to be secure against quantum adversaries.

What is Lattice?

- A lattice can be thought of as any regularly spaced grid of points stretching out to infinity.



Definition of Lattice

Let $B = [\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k]$ be k linearly independent vectors in $\mathbb{R}^n$. Then,

$$\Lambda = \mathcal{L}(B) := \left\{ \sum_{i=1}^{k} x_i \vec{b}_i : x_i \in \mathbb{Z} \right\}$$

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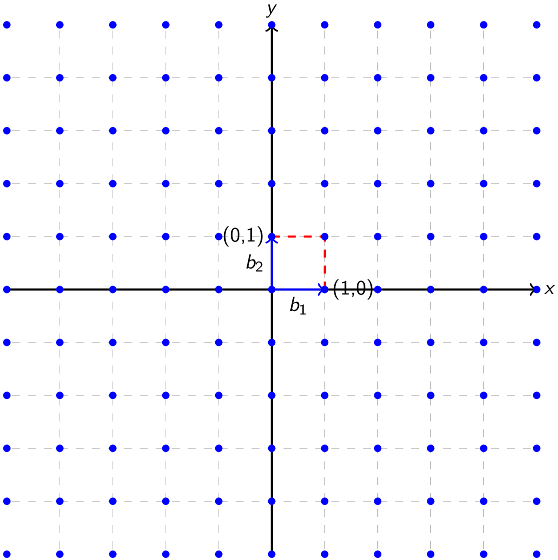
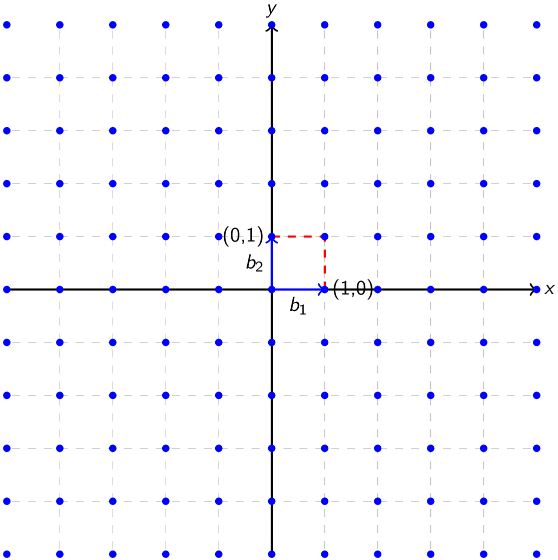
- (n, k) is called the (dimension, rank) of the lattice
- If $n = k$, then $\mathcal{L}(B)$ is called the *full rank lattice*.

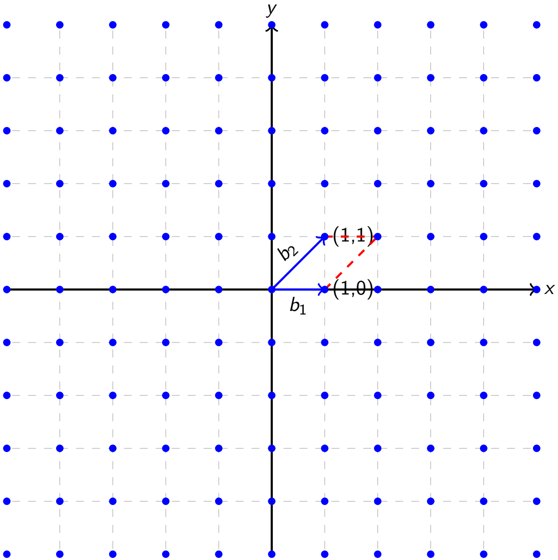
Example of Lattice



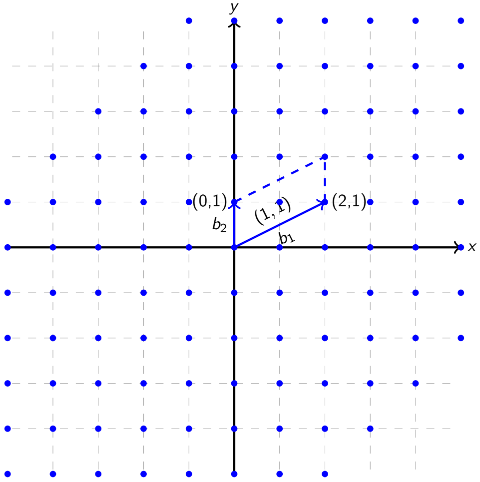
Is a Lattice Basis Unique ?



Another Basis of $\mathbb{Z}^2$



Does It Generate $\mathbb{Z}^2$?



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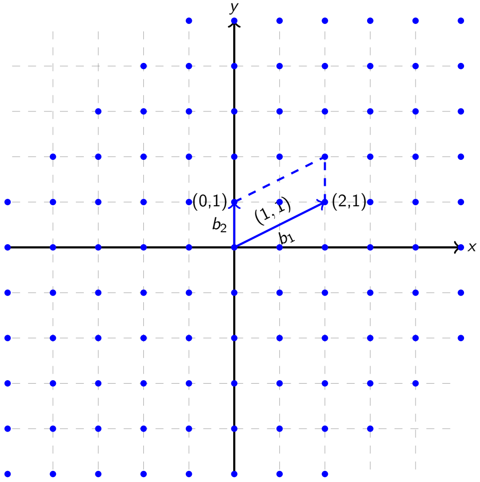

What is the criteria that two basis generate the same lattice ?

Transformation of Basis

- swap (i, j), i.e., $\vec{b}_i \leftrightarrow \vec{b}_j$
- invert (i), i.e., $\vec{b}_i \leftrightarrow -\vec{b}_i$
- add (i, j, c), i.e., $\vec{b}_i \leftarrow \vec{b}_i + c\vec{b}_j$ for some non-zero $c \in \mathbb{Z}$

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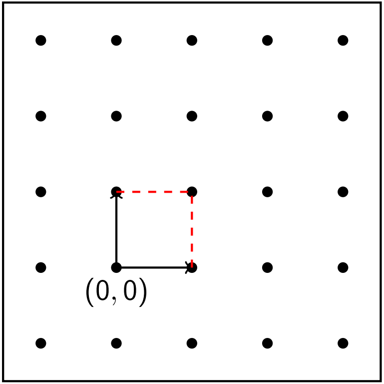

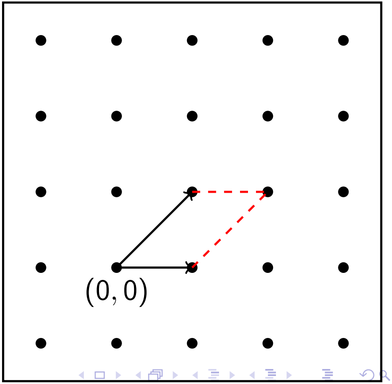

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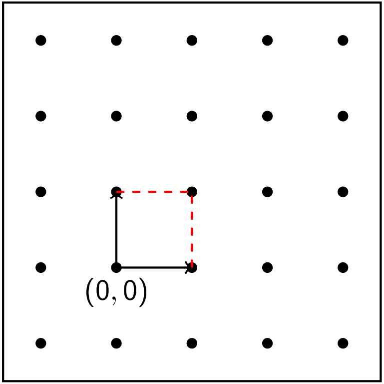

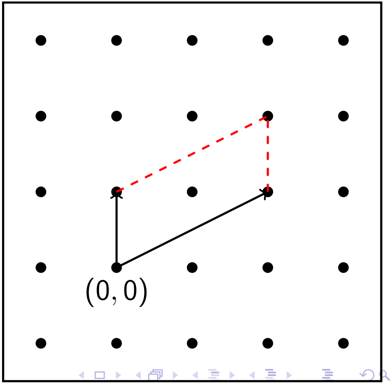

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A matrix $U \in \mathbb{Z}^{n \times n}$ is called a unimodular matrix if $U \in \mathrm{GL}(n, \mathbb{Z})$

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Fundamental Region and Fundamental Parallelepiped

Let $B \in \mathbb{R}^{n \times k}$ be a basis. We define fundamental parallelepiped corresponding to the lattice $\mathcal{L}(B)$ as $\mathcal{P}(B) := \{B.x : x \in \mathbb{R}^k, 0 \leq x_i < 1\}$.

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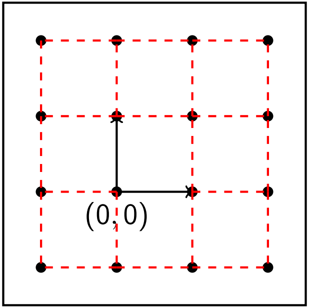
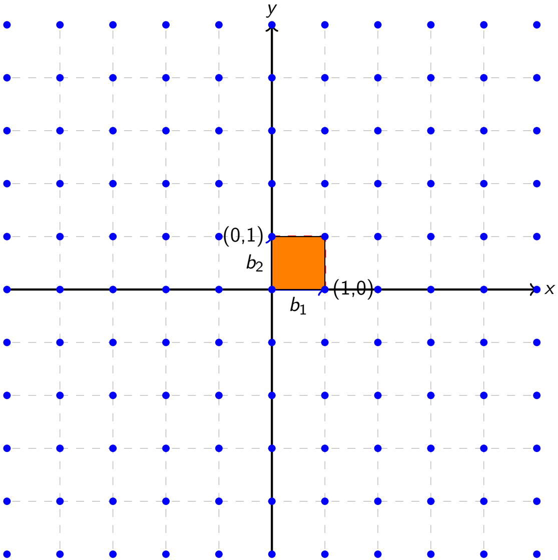
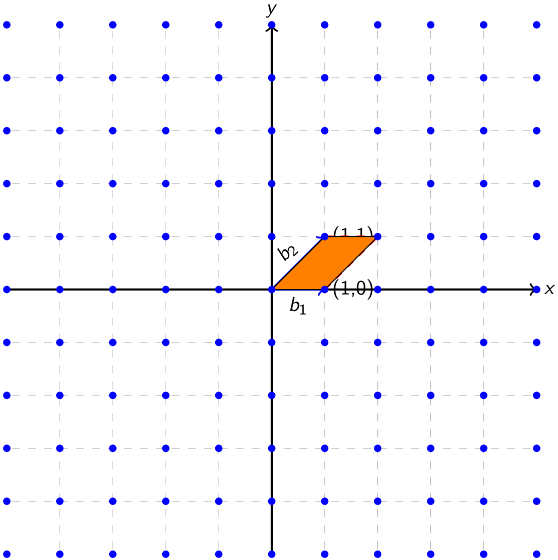
Property: Fundamental parallelepiped tiles the span of B.



Examples of Fundamental Parallelepiped



Fundamental Parallelepiped Changes with Basis



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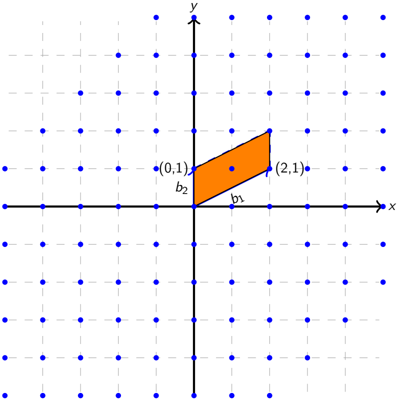

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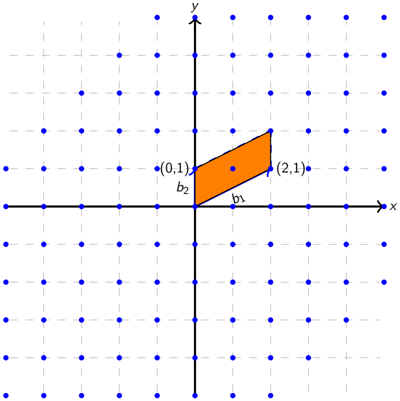

Given a lattice Λ and a set of linearly independent lattice vectors from Λ, when does it generate the same lattice ?

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- $\because \vec{x}_1, \ldots, \vec{x}_n$ are linearly independent, $y_i - \lfloor y_i \rfloor = 0 \iff y_i = \lfloor y_i \rfloor$.

Determinant of a Lattice

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Note: $\det(\Lambda) \propto \frac{1}{\text{density}(\Lambda)}$.

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$$\pi_i(x) = x - \sum_{j=1}^{i-1} \frac{\langle x, b_j^* \rangle}{\langle b_j^*, b_j^* \rangle} b_j^*$$

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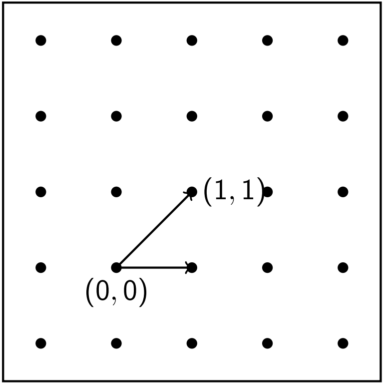
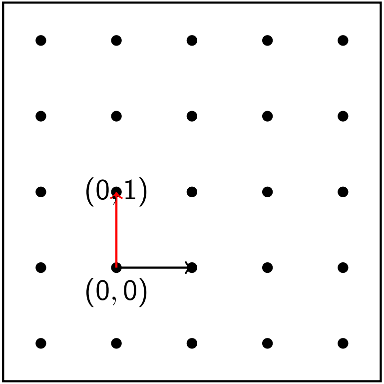
The GSO of a sequence of vectors $B = (b_1, b_2, \ldots, b_n)$ is the sequence $B^* = (\pi_1(b_1), \pi_2(b_2), \ldots, \pi_n(b_n))$.

Example of GSO





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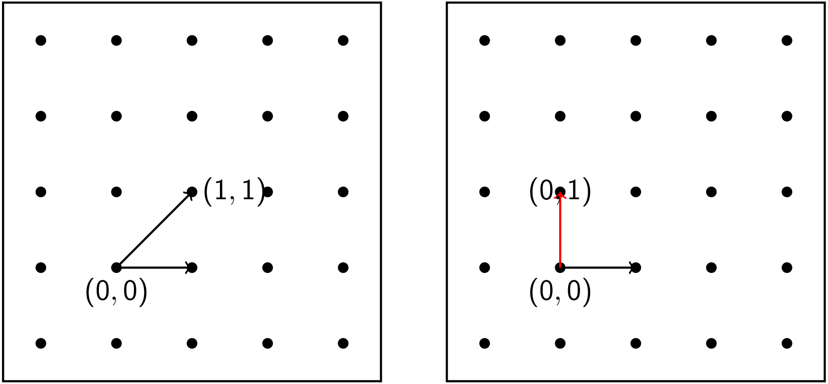
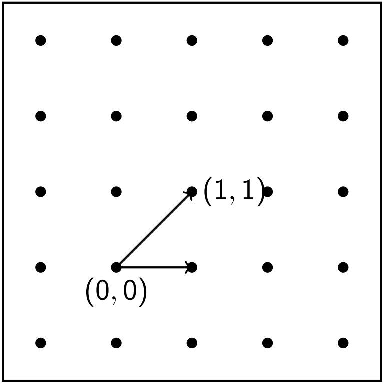
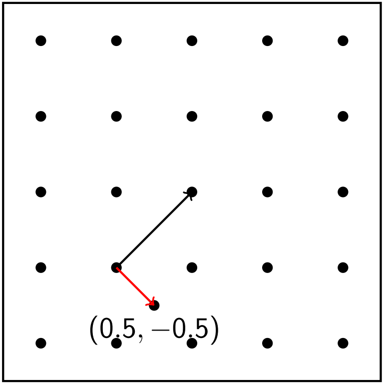


Both the basis generate the same lattice

Example of GSO





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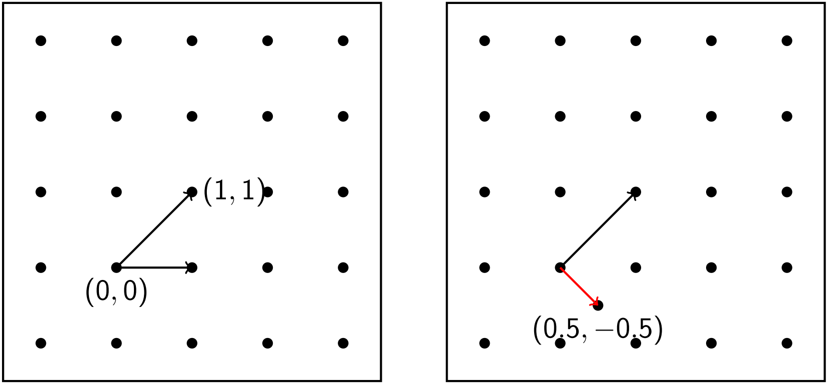

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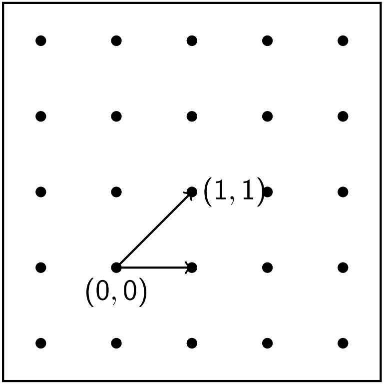

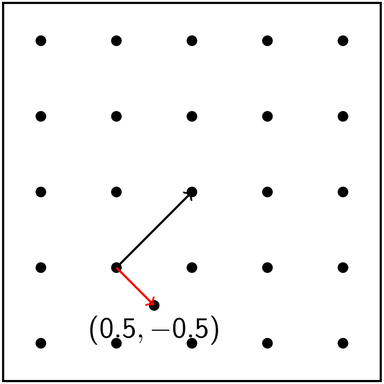

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In general, $\mathcal{L}(B) \neq \mathcal{L}(B^*)$

Determinant of a Lattice and Gram-Schmidt Vectors

Lemma

Ler B be a basis and B^* be its GSO. Then, $\text{vol}(\mathcal{P}(B)) = \prod_{i=1}^{n} \|b_i^*\|$

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- Consider $n = 2$. Let $B = (b_1, b_2)$.
- $\mathcal{P}(B)$ is a parallelogram with sides b_1 and b_2
- The area of the parallelogram is $\|b_1\| \times \|b_2^*\|$
- Let it be true for dimension $n - 1$
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Since $\|b_i^*\| \leq \|b_i\|$, we have $\text{vol}(\mathcal{P}(B)) \leq \prod_i \|b_i\|$.

Minimum Distance of a Lattice

For any lattice Λ, the minimum distance of Λ is the smallest distance between any two lattice points, i.e.,

$$\lambda_1(\Lambda) := \inf\{\|\vec{x} - \vec{y}\| : \vec{x}, \vec{y} \in \Lambda, \vec{x} \neq \vec{y}.\}$$

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Since, Λ is a discrete additive subgroup of $\mathbb{R}^n$, we can alternatively defined it the minimum norm of a non-zero lattice vector, i.e.,

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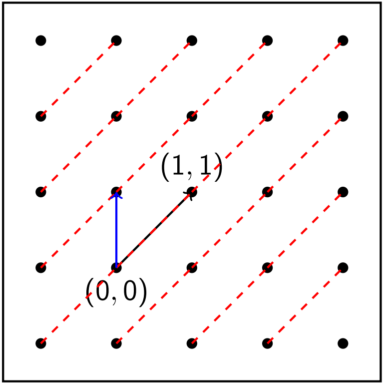

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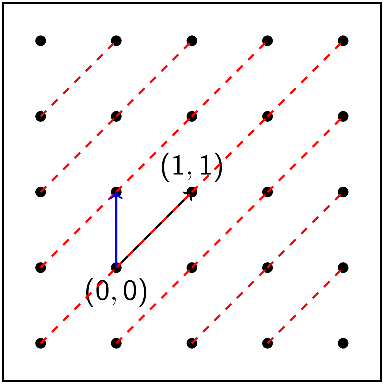

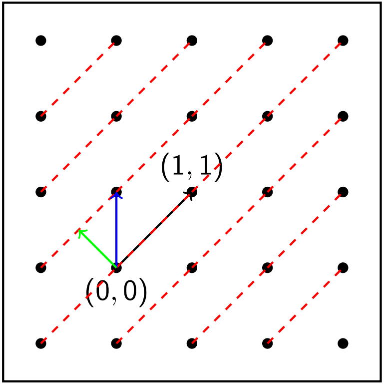

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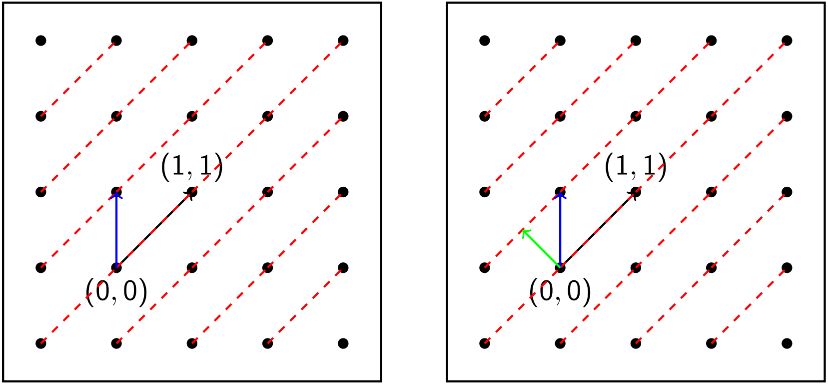

Any discrete subgroup of $\mathbb{R}^n$ is a lattice

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- A consequence is that the ball centered at 0 and radius k will contain finitely many lattice points.
- Consider a closed ball $B(0, 2\lambda_1)$. This contains finitely many lattice points. Then by definition of λ_1 , there is at least one lattice point of length λ_1.

Minkowski's Bound on λ_1

For any n-dimensional lattice Λ, $\lambda_1(\Lambda) \leq \sqrt{n}\det(\Lambda)^{1/n}$

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(**Tightness of the Bound:**) Let D be a large integer.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}$$

Note that, $\lambda_1(\mathcal{L}(B)) = 1$ but, the result says $\lambda_1(\mathcal{L}(B)) \leq \sqrt{2D}$

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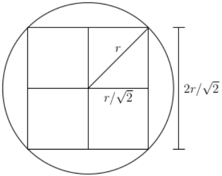

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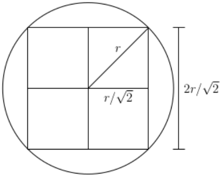

- $\text{vol}(B(0, \lambda_1)) \geq \left(\frac{2\lambda_1}{\sqrt{n}}\right)^n$
- $\text{vol}(B(0, \lambda_1)) \leq 2^n \det(\mathcal{L}(B))$ – (MCB Theorem)

Proof of Minkowski's Convex Body Theorem

Minkowski's Convex Body Theorem

Let Λ be a full-rank lattice of dimension n. If $S \subseteq \mathbb{R}^n$ is a centrally symmetric and convex body of volume $\text{vol}(S) > 2^n \det(\Lambda)$, then S contains a non-zero lattice point.

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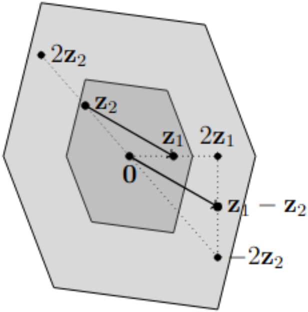

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Let B be a basis and Λ be an n-dimensional full rank lattice. Let $S \subseteq \text{span}(\Lambda)$ be a measureable set such that $\text{vol}(S) > \det(\Lambda)$. Then, there exists two points z_1, z_2 such that $z_1 - z_2 \in \Lambda$.

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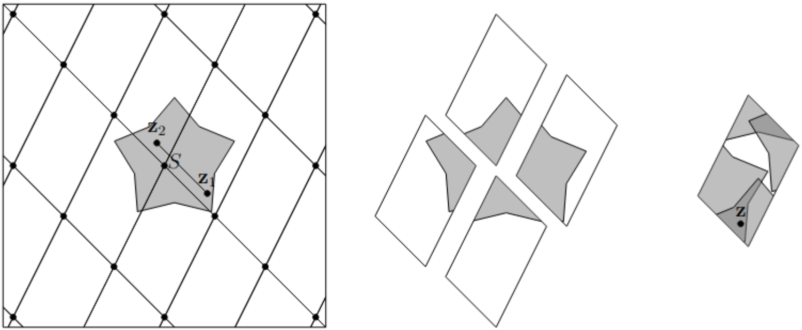

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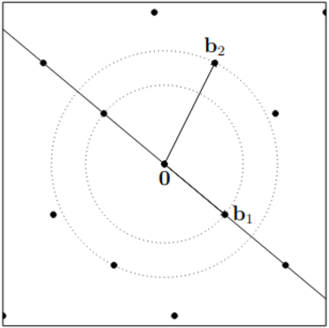

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Minkowski's Second Theorem

For an n-dimensional lattice Λ, $(\prod_i(\lambda_i))^{1/n} \leq \sqrt{\gamma_n}(\det(\Lambda))^{1/n}$, where γ_n is the Hermite constant.

Sublattice

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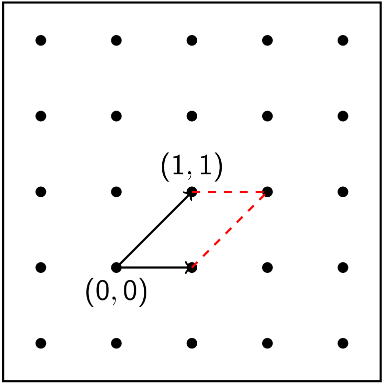

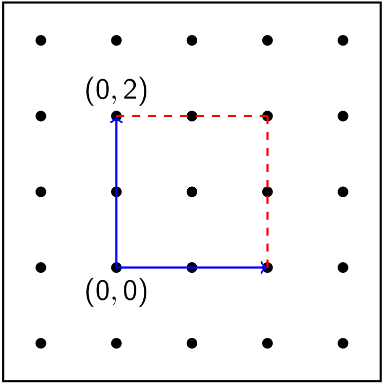

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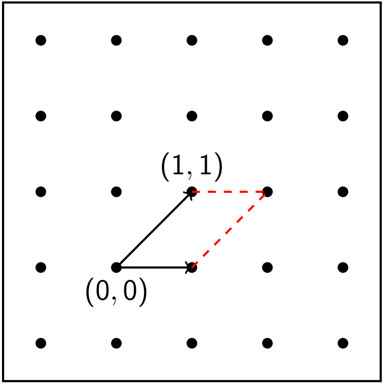

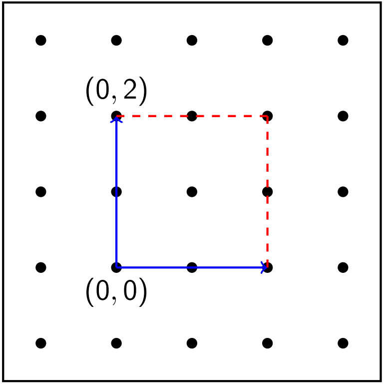

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = B \cdot \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

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Note that, $\frac{\det(\mathcal{L}(B))}{\det(\mathcal{L}(C))}$ should be an integer.

Group Theoretic View of Lattice and Sublattice

Defn: An n-dimensional Lattice Λ is a discrete additive subgroup of $\mathbb{R}^n$

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It can be shown that the two definitions of lattice are equivalent

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- Define an equivalence relation $\equiv_{\Lambda'}$ over Λ as $x \equiv_{\Lambda'} y$ iff $x - y \in \Lambda'$
- For $x, x' \in [x]$ and $y, y' \in [y]$, $[x] + [y] = [x + y]$ which is defined as $x + x' \equiv_{\Lambda'} y + y'$
- The collection of equivalence classes along with the operation '$+$' constitute a group, called **quotient group**, denoted as Λ/Λ'
- Each equivalence class $[x]$ is a set $x + \Lambda'$ which is known as a **coset**, where $x + \Lambda = \{v \in \Lambda : v - x \in \Lambda'\}$
- Note that $\Lambda = \bigcup_{x \in \Lambda/\Lambda'} x + \Lambda'$

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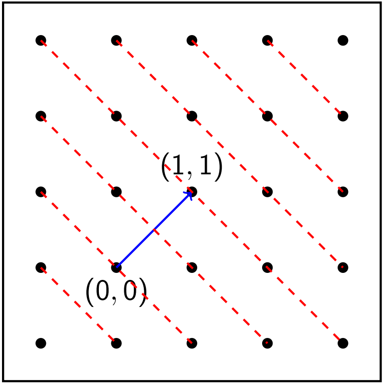

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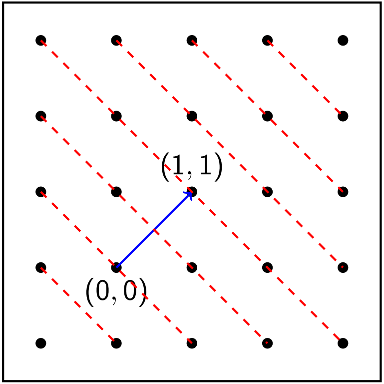

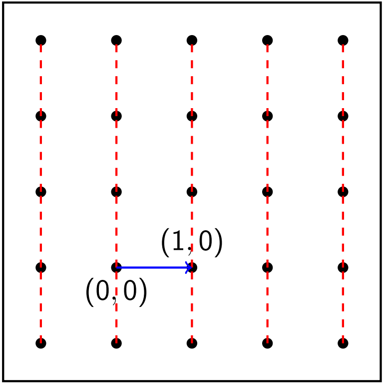

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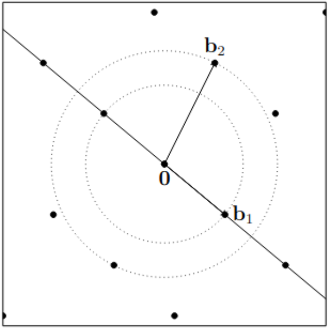
You can prove all the above properties at home!

Transference Theorem Related to Successive Minima



Goal: Consider Λ with successive minima $\lambda_1, \ldots, \lambda_n$ and $\widehat{\Lambda}$ be the dual lattice of Λ with successive minima $\widehat{\lambda_1}, \ldots, \widehat{\lambda_n}$. Can we transfer knowledge from $\lambda_1, \ldots, \lambda_n$ to $\widehat{\lambda_1}, \ldots, \widehat{\lambda_n}$?

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Remark

Using the stronger version of Minkowski's Theorem, one has

$$\lambda_1 \cdot \widehat{\lambda_1} \leq \gamma_n.$$

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Banaszczyk Transference Theorem.

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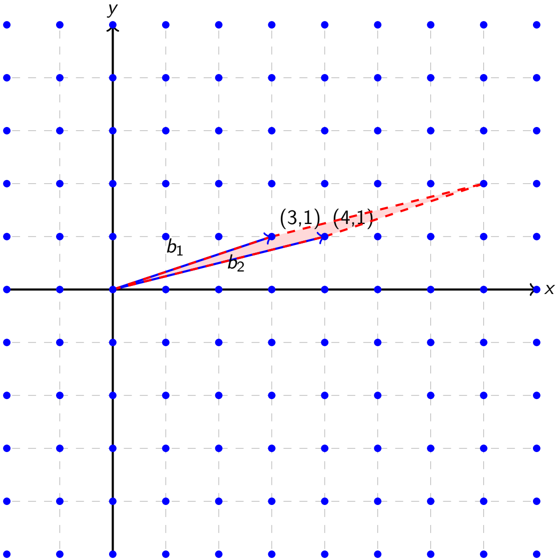

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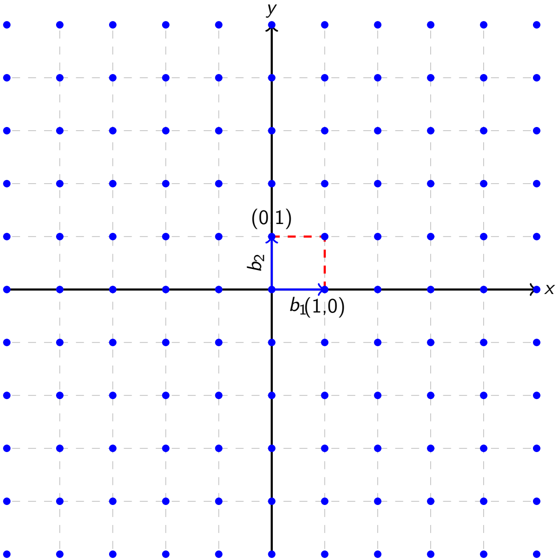

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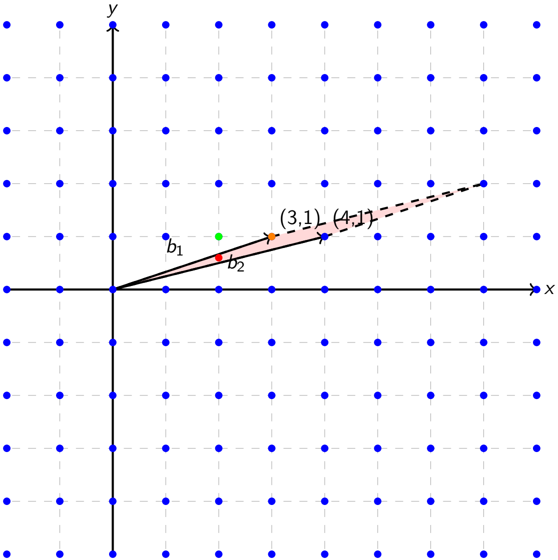

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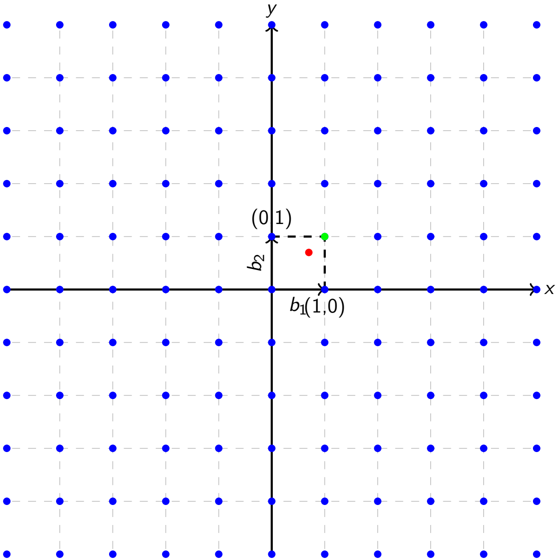

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Thank You!