0, 1

$5V \to 1$

$0V \to 0$

$> 2.5 \to 1$

$\leq 2.5 \to 0$

✓ Superconducting Qubits

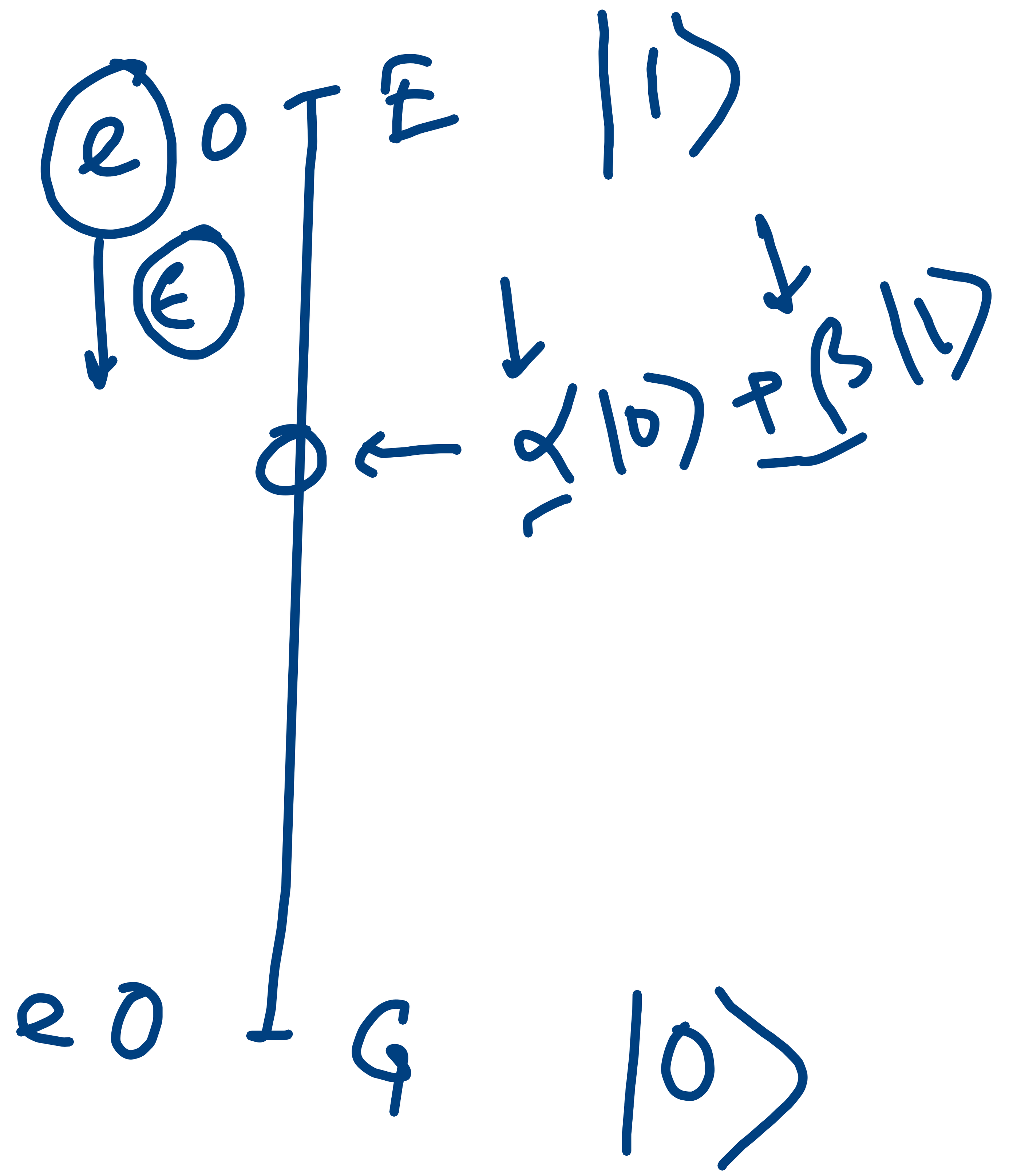
Trapped Ion Qubits

Spin Qubits

⋮

$\alpha, \beta \in \mathbb{C}$

✓ $|\alpha|^2 + |\beta|^2 = 1$



$| \rangle \rightarrow$ Qubit

$\langle | \rightarrow$ Conjugate transpose of $| \rangle$

$$\langle \psi | . | \psi \rangle \Rightarrow |\alpha|^2 + |\beta|^2 = 1$$

$$\Downarrow$$

$$\underline{\langle \psi | \psi \rangle} = 1$$

$\downarrow$

inner product.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\underline{|0\rangle} \otimes \underline{|0\rangle} \equiv |00\rangle = |0\rangle|0\rangle$$

$\downarrow$

tensor product

$$|01\rangle, |10\rangle, |11\rangle$$

$$\underline{|\psi_1\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle}$$

$$a, b, c, d \in \mathbb{C}$$

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$$

$$|\psi_1\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$a=1 \Rightarrow |\psi_1\rangle = |00\rangle.$

$\Downarrow$

$\underline{|0\rangle} \otimes \underline{|0\rangle}$

$\underline{|\psi_1\rangle} \neq |\phi_1\rangle \otimes |\phi_2\rangle$

↓ separable state

$$\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} \otimes \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix} \rightarrow \begin{pmatrix}\alpha_1\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\\ \beta_1\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\alpha_1\alpha_2\\ \alpha_1\beta_2\\ \beta_1\alpha_2\\ \beta_1\beta_2\end{pmatrix}$$

$ad \neq bc \rightarrow$ entangled

$ad = bc \rightarrow$ separable

→ Entangled state

$$\underline{\alpha|00\rangle + \beta|11\rangle} = \begin{pmatrix}\alpha\\ 0\\ 0\\ \beta\end{pmatrix} \quad \cdot \quad \alpha|00\rangle + \beta|01\rangle = \underline{|0\rangle} \otimes \underline{(\alpha|0\rangle + \beta|1\rangle)}$$

1 Qubit → (2×1) col. vec

2 " → (4×1) " "

n " → $(2^n\times1)$ " "

Bell states

$$\boxed{\begin{array}{l}\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\\ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\end{array}}$$

$|\psi_1\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

$a=1 \Rightarrow |\psi_1\rangle = |00\rangle.$

$\Downarrow$

$\underline{|0\rangle} \otimes \underline{|0\rangle}$

$\underline{|\psi_1\rangle} \neq |\phi_1\rangle \otimes |\phi_2\rangle$

$\downarrow$ separable state

$\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} \otimes \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}$

$\begin{pmatrix}\alpha_1\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\\ \beta_1\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\alpha_1\alpha_2\\ \alpha_1\beta_2\\ \beta_1\alpha_2\\ \beta_1\beta_2\end{pmatrix}$

$ad \neq bc \rightarrow$ entangled

$ad = bc \rightarrow$ separable

$\rightarrow$ Entangled state

Bell states

$$\boxed{\begin{array}{l}\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)\\ \frac{1}{\sqrt{2}}\left(|00\rangle-|11\rangle\right)\\ \frac{1}{\sqrt{2}}\left(|01\rangle+|10\rangle\right)\\ \frac{1}{\sqrt{2}}\left(|01\rangle-|10\rangle\right)\end{array}}$$

$\underline{\alpha|00\rangle + \beta|11\rangle} = \begin{pmatrix}\alpha\\ 0\\ 0\\ \beta\end{pmatrix}$

$\rightarrow$

$\alpha|00\rangle + \beta|01\rangle$

$= \underline{|0\rangle} \otimes \underline{\left(\alpha|0\rangle + \beta|1\rangle\right)}$

1 Qubit $\rightarrow (2\times 1)$ col. vec

2 " $\rightarrow (4\times 1)$ " "

n " $\rightarrow (2^n\times 1)$ " "

GHZ state

$$\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

W state $\frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$

✓ $\frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\right)$

$$U|\psi\rangle \rightarrow |\psi'\rangle$$

$$(U|\psi\rangle)^{\dagger} = \langle\psi|\underline{U^{\dagger}U}|\psi\rangle$$

$\dagger \rightarrow$ daggar = conjugate transpose

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{|0\rangle}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \overset{|1\rangle}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

bit flip gate.

$$\overset{|1\rangle}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \overset{|0\rangle}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\langle\psi|\psi\rangle = 1$$

$$U^{\dagger}U = I = UU^{\dagger} = \text{Unitary.}$$

$$\langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1$$

$$Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \overset{|0\rangle}{\begin{pmatrix}1\\ 0\end{pmatrix}} = \begin{pmatrix}1\\ 0\end{pmatrix} = |0\rangle$$

$$\overset{|1\rangle}{\begin{pmatrix}0\\ 1\end{pmatrix}} = \begin{pmatrix}0\\ -1\end{pmatrix} = -|1\rangle$$

phase flip gate.

$$Y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = iXZ$$

$$X(\alpha|0\rangle + \beta|1\rangle) = \alpha\, X|0\rangle + \beta\, X|1\rangle = \alpha|1\rangle + \beta|0\rangle$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$\text{CNOT}\ |a\rangle|b\rangle \longrightarrow |a\rangle|a\oplus b\rangle$$

$|a\rangle$: control qubit

$|b\rangle$: target qubit

$$\text{CNOT}\ \underline{(\alpha|0\rangle + \beta|1\rangle)\ |0\rangle}$$

$$\text{CNOT}\ (\alpha|00\rangle + \beta|10\rangle)$$

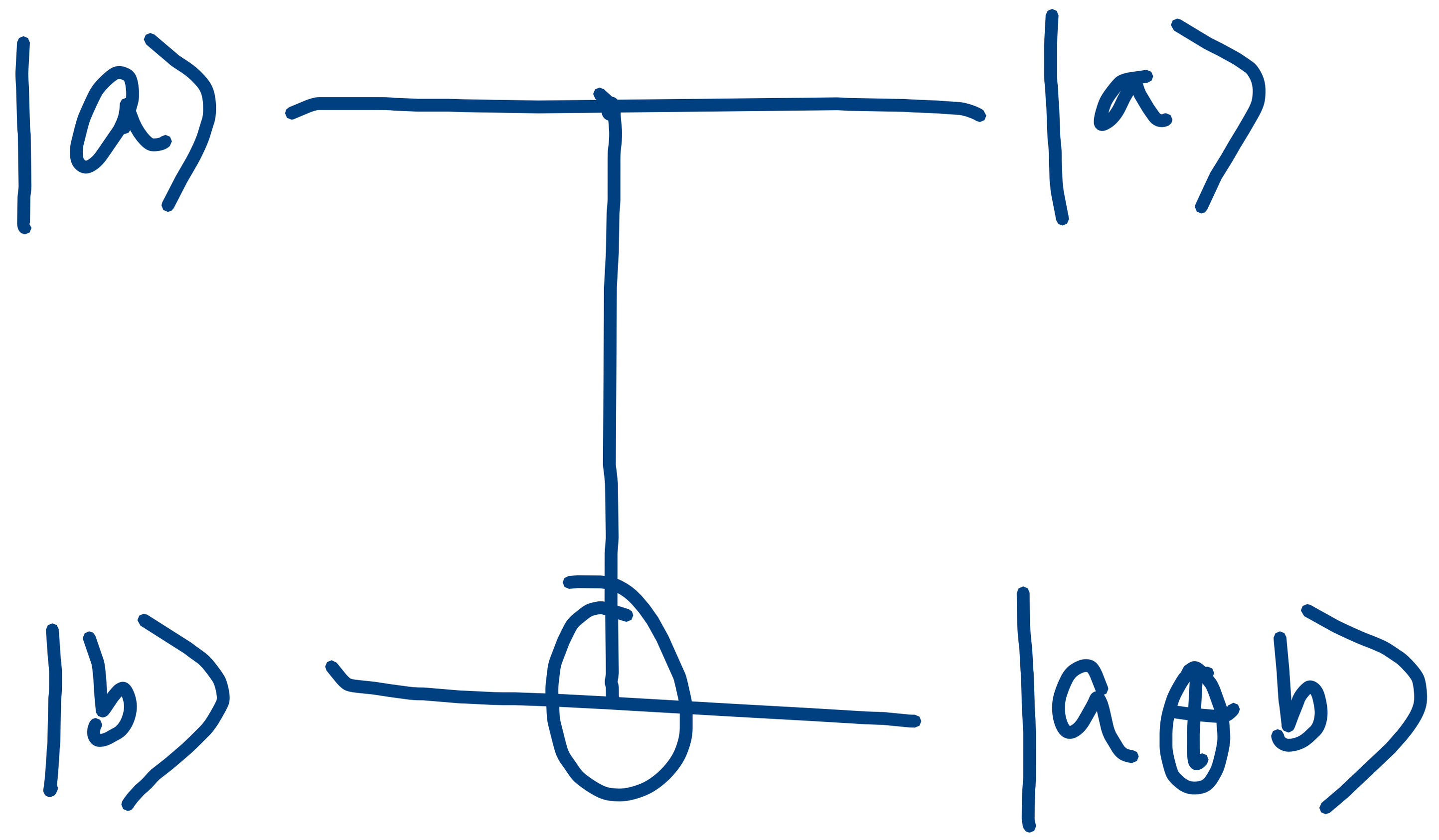
$$\alpha|00\rangle + \beta|11\rangle$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

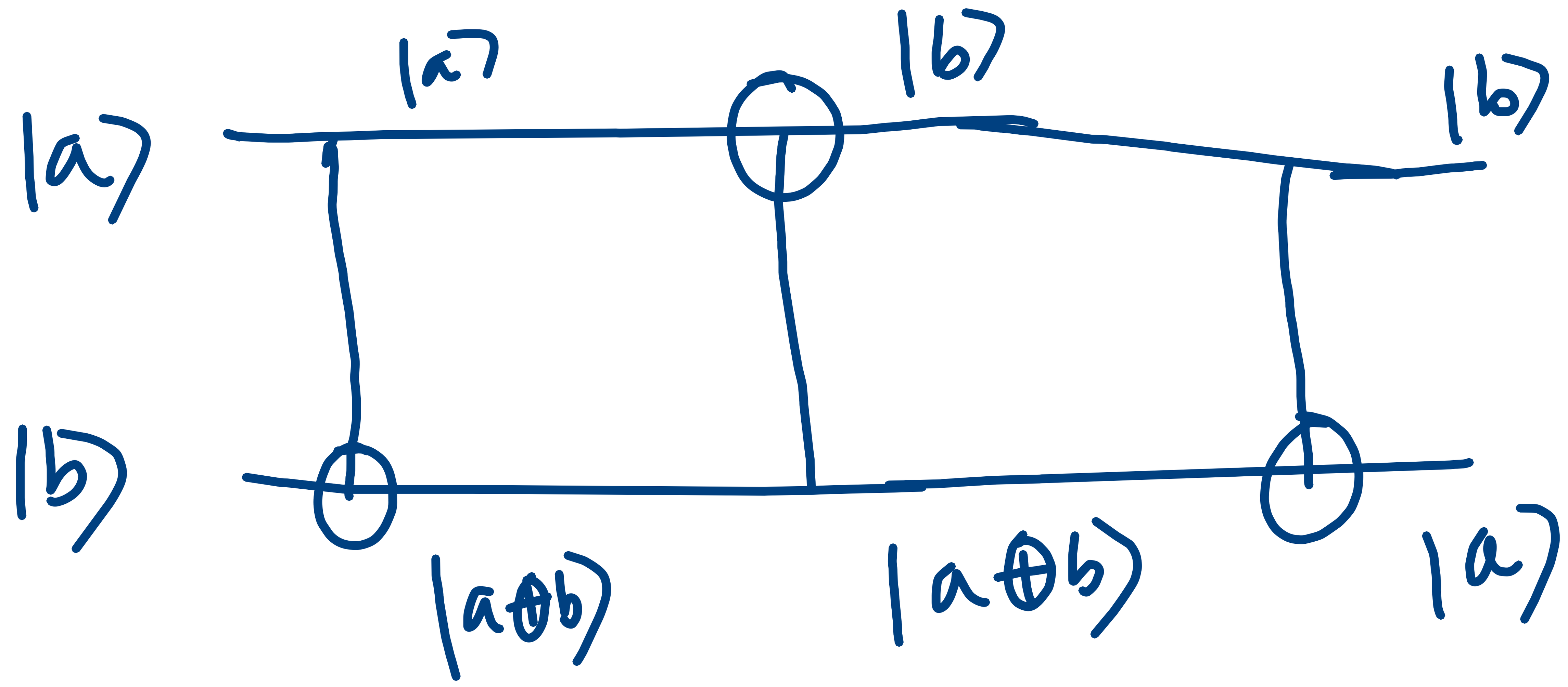
$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$CNOT\ \underline{|a\rangle}\,|b\rangle \rightarrow |a\rangle\,|a \oplus b\rangle$

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}} \end{pmatrix}$$



$SWAP\ |a\rangle\,|b\rangle \rightarrow |b\rangle\,|a\rangle$



3 Qubit

Toffoli Gate $|a\rangle\,|b\rangle\,|c\rangle \rightarrow |a\rangle\,|b\rangle\,|c \oplus ab\rangle$

$\{|0\rangle, |1\rangle\}$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi\rangle = |0\rangle = \cos\theta|\phi\rangle + \sin\theta|\phi^{\perp}\rangle$

$\{|0\rangle, |1\rangle\}$

$\{|v_1\rangle, |v_2\rangle\}$

$|0\rangle \to |\alpha|^2$

$|1\rangle \to |\beta|^2$

$\{|+\rangle, |-\rangle\}$

$$\downarrow$$

$\Pr(\text{outcome } |v_i\rangle) = |\langle v_i|\psi\rangle|^2$

$\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

$H|0\rangle \to \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$

$\{|+\rangle, |-\rangle\}$

$|\phi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$

$\{|\phi\rangle, |\phi^{\perp}\rangle\}$

$|\phi^{\perp}\rangle = \sin\theta|0\rangle - \cos\theta|1\rangle$

$$\underbrace{\alpha|00\rangle+\beta|11\rangle}\neq|\psi_1\rangle\otimes|\psi_2\rangle$$

$$\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)$$

A B A B

a	b
0	0
0	1
1	0
1	1

$a=1$, $A\to Z$

$a=0$, $A\to I$

$b=1$, $A=X$

$b=0$, $A=I$

1 qubit communication

Bob $\to$ CNOT

$\to$ H on first qubit

measure in $\{|0\rangle,|1\rangle\}$ basis

ab

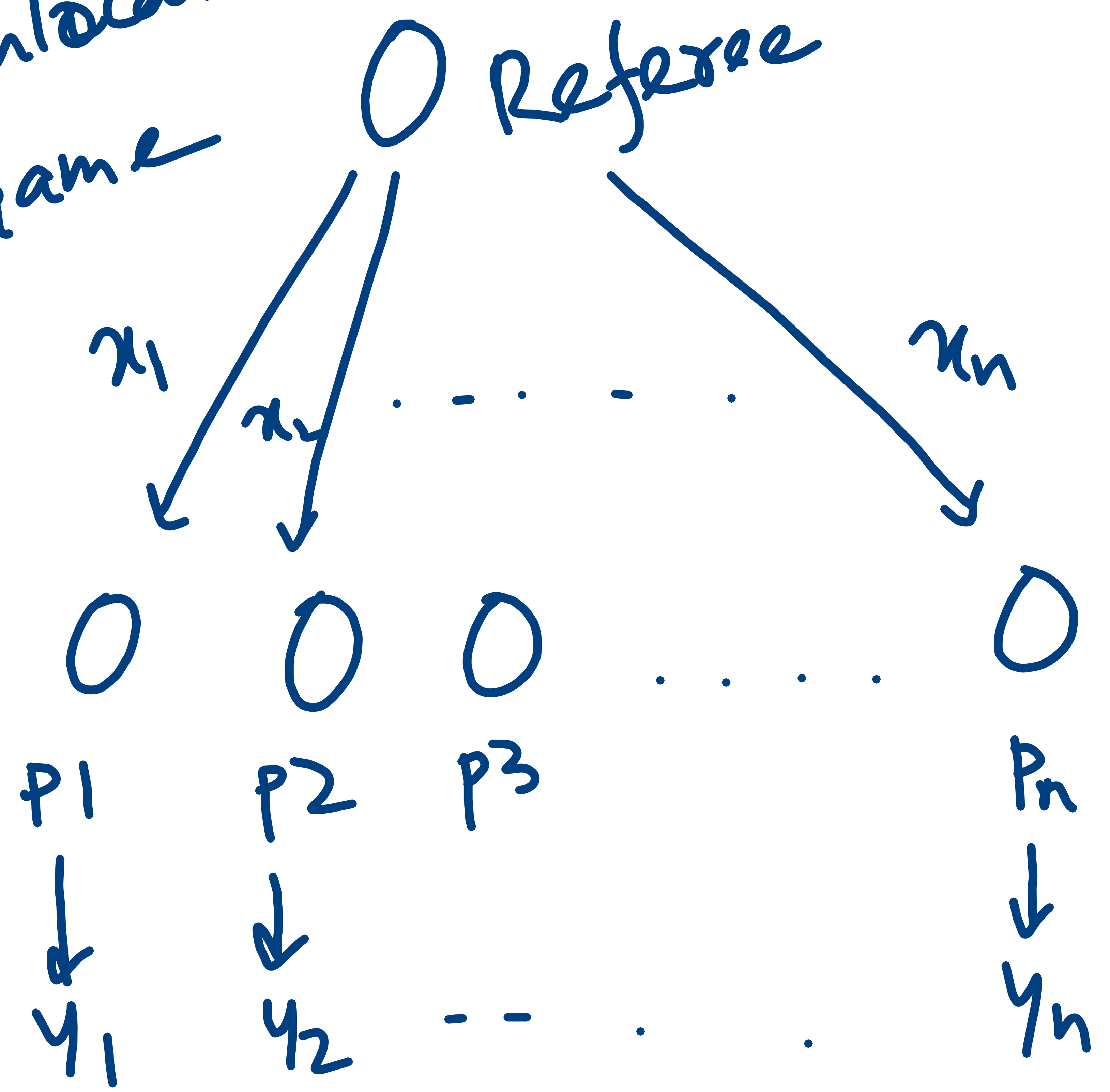
$00\to\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\xrightarrow{\text{CNOT}}\frac{1}{\sqrt{2}}(|00\rangle+|10\rangle)=\frac{(|0\rangle+|1\rangle)}{\sqrt{2}}|0\rangle=|+\rangle|0\rangle\xrightarrow{H}\boxed{|00\rangle}$

$01\to\frac{1}{\sqrt{2}}(|10\rangle+|01\rangle)\xrightarrow{\text{CNOT}}\frac{1}{\sqrt{2}}(|11\rangle+|01\rangle)=|+\rangle|1\rangle\to\boxed{|01\rangle}$

$10\to\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)\xrightarrow{\text{CNOT}}\frac{1}{\sqrt{2}}(|00\rangle-|10\rangle)=|-\rangle|0\rangle\xrightarrow{H}\boxed{|10\rangle}$

$11\to\frac{1}{\sqrt{2}}(|10\rangle-|01\rangle)\xrightarrow{\text{CNOT}}\frac{1}{\sqrt{2}}(|11\rangle-|01\rangle)=-|-\rangle|1\rangle\xrightarrow{H}\boxed{-|11\rangle}$

Nonlocal Game



$x_i \in \{0,1\}$

$y_i \in \{0,1\}$

$f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$

$\rightarrow 0, 1, x, \bar{x}$

A → i/p 0 → o/p a_0

→ i/p 1 → o/p a_1

B → i/p 0 → o/p b_0

i/p 1 → o/p b_1

CHSH Game

i/p	
00	$a_0 \oplus b_0 = 0$
01	$a_0 \oplus b_1 = 0$
10	$a_1 \oplus b_0 = 0$
11	$a_1 \oplus b_1 = 1$

$x \times y = a \oplus b$

$A \qquad B$

$$\frac{1}{\sqrt{2}}\left(|\underset{A}{0}\underset{B}{0}\rangle + |\underset{A}{1}\underset{B}{1}\rangle\right)$$

$A = x = 0 \rightarrow \{|0\rangle, |1\rangle\}$ basis

$= x = 1 \rightarrow \{|+\rangle, |-\rangle\}$ basis

$B = y = 0 \rightarrow$ measure in $\frac{\pi}{8}$ rotated basis

$y = 1 \rightarrow$ " " $-\frac{\pi}{8}$ " "

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\{\cos\theta|0\rangle + \sin\theta|1\rangle, \; -\sin\theta|0\rangle + \cos\theta|1\rangle\}$

$\text{win} = \cos^2\frac{\pi}{8} \approx 0.85$

Tsirelsen Bound

$$U|\psi\rangle|0\rangle \to |\psi\rangle|\psi\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\begin{cases} U|0\rangle|0\rangle \to |0\rangle|0\rangle \\ U|1\rangle|0\rangle \to |1\rangle|1\rangle \end{cases}$$

$$\underline{U(\alpha|0\rangle + \beta|1\rangle)|0\rangle} \to \underline{(\alpha|0\rangle+\beta|1\rangle)(\alpha|0\rangle+\beta|1\rangle)}$$

$$U(\alpha|00\rangle + \beta|10\rangle)$$

$$\alpha|00\rangle + \beta|11\rangle \neq (\alpha|0\rangle+\beta|1\rangle)(\alpha|0\rangle+\beta|1\rangle)$$

Can't copy the state of any unknown qubit

$|0\rangle \qquad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad \{|v_1\rangle, |v_2\rangle\} \qquad pr(|v_1\rangle), pr(|v_2\rangle)$

$|\psi\rangle = |0\rangle / |+\rangle \qquad |\psi\rangle \rightarrow |v_1\rangle$

$0 = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$

$|0\rangle \qquad pr(|1\rangle) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$

1. No Cloning
2. Indistinguishability of non orthogonal states

$\{\cos\theta|0\rangle + \sin\theta|1\rangle, |0\rangle\} \Rightarrow \frac{\sin^2\theta}{2}$ (Projective)

$(1-\cos\theta)$ (POVM)

Quantum Key Distribution (QKD)

✓ 1. Information Reconciliation

✓ 2. Privacy Amplification.

$$A = 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \rightarrow \text{part of this will generate raw key}$$

$$\rightarrow B = 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \rightarrow \text{Basis Choice}$$

$$|+\rangle \quad |0\rangle \quad \underbrace{|1\rangle \quad |1\rangle}_{\{|+\rangle, |-\rangle\}} \quad |+\rangle \quad \underbrace{|+\rangle \quad |+\rangle}_{\{|0\rangle, |1\rangle\}} \quad |-\rangle \xrightarrow{\text{Quantum channel}} \text{Bob}$$

Adversary

Measure in Randomly chosen basis.

$$|+\rangle \quad |0\rangle \quad |0\rangle \quad |-\rangle \rightarrow \text{Raw key}$$

A	B	
0	0	$\rightarrow |0\rangle$
0	1	$\rightarrow |+\rangle$
1	0	$\rightarrow |1\rangle$
1	1	$\rightarrow |-\rangle$

$B = 0 \rightarrow \{|0\rangle, |1\rangle\}$

$B = 1 \rightarrow \{|+\rangle, |-\rangle\}$