A new global measures of teleportation in Quantum Networks

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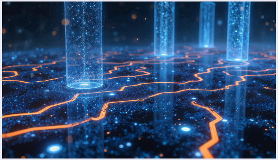
Outline of the talk

- Background and Motivation
- Resource State for Teleportation
- Repeaters and Entanglement Swapping
- Average of Maximum Teleportation Fidelity
- Network without loops
- Network with loops
- Impact and Conclusions
- Thank You

Background and Motivation

What is a **network**?
A network is a system made up of connected components. This can be a group of computers, devices, or people. They are connected for communication or sharing resources.
Network science is a field that studies the structure, dynamics of **complex networks**.

Complex networks



Types of complex network:
Telecommunication networks, computer networks, biological networks, cognitive and semantic networks, and social networks. Distinct elements are represented by nodes and the connections between them as links.

Background and Motivation

Why Topology matters when Network grows

- Defines Structure: The way nodes are arranged (star, chain, mesh, scale-free) shapes the network's behavior.
- Affects Efficiency: Determines how quickly information or resources can flow.
- Influences Robustness: Some topologies resist failures better than others.
- Shapes Growth: Topology guides whether hubs, clusters, or bottlenecks emerge.

Parameters :

Local Parameters: Degree, Clustering Coefficient, Betweeness Centrality. etc .

Global parameters: Average Path Length, Network Diameter . etc

Background and Motivation

Importance of Global Parameters

- Holistic view: Captures how network behave, not just local nodes, source and target
- Efficiency: Reveal how well information flow globally.
- Robustness: Ability of networks to handle failures.
- Scalability: Predict the performance once networks grow larger.

Background and Motivation

Stages of Quantum Network

- **Pre Quantum Network (Trusted Repeater Network) :** If the nodes are directly connected or connected by trusted repeater we can do QKD, No quantum information exchange
- **Proto-quantum networks:** End to end delivery of qubits are possible. Key distribution between any two noes. Distribution of entanglement between arbitrary nodes in the network is possible
- **Advanced Quantum Network:** Possible to keep quantum information in a quantum memory for a certain time. This allows to carry out teleportation across network. **This work is based on Advanced Network (Liberty of Theoreticians)**

Background and Motivation

Quantum Network

Q: Do Global Parameters matter in Quantum Networks?

How do a define Global parameter in a Quantum Network? There has to be a task that is essentially quantum. There has to be a measure which is quantum

Is teleportation one of such task? $\rightarrow$ **Yes**

Is fidelity one such measure $\rightarrow$ **Yes**

Background and Motivation

Motivation:

- Define a **Global Parameter** in Quantum Network to understand the network's **capability in distributed task**.
- The task here we choose here is **Teleportation**. There can be other task.
- Aim to characterize the resourcefulness of a network in terms of **Quantum Advantage**.
- Eventually **ranking** the network in terms of their abilities for a particular choice of nodes and input parameter.
- Understanding the quantum network from the perspective of **network science**.

Resource State for Teleportation

Teleportation: Since teleportation is one such task, lets talk about teleportation$\rightarrow$
We all know that the teleportation is the process of sending unknown quantum information from one location to another with the help of shared entangled state [1] .

Q: What are the resource states for teleportation?

The shared entangled state with which we can do better than the **classical limit** gives us quantum advantage and hence resource. The **classical limit** of teleportation (without entangled state)for two qubit state is $\frac{2}{3}$.
Note: Given an entangled state it **may not be useful for teleportation**.

[1]R. Horodecki, M. Horodecki, and P. Horodecki, Teleportation, Bell's inequalities and inseparability, Phys. Lett. A 222, 21 (1996).

Resource State for Teleportation

Two Qubit Density Matrices:

$$\rho = \frac{1}{4}\left[I_4 + \sum_i r_i \sigma_i \otimes I + \sum_j s_j I \otimes \sigma_j + \sum_{i,j} t_{ij} \sigma_i \otimes \sigma_j \right], \qquad (1)$$

Teleportation Fidelity:

$$F_{\max}(\rho) = \frac{1}{2}\left[1 + \frac{1}{3} N(\rho)\right], \qquad N(\rho) = \mathrm{Tr}\left(\sqrt{T^\dagger T}\right) \qquad (2)$$

States with $N(\rho) > 1$ or equivalently $F_{\max}(\rho) > \frac{2}{3}$ provide quantum advantage-**Resource** [2]. **Our Model :**

$$\rho_{\text{wer}} = \frac{1-p}{4}\mathbb{I} + pP(|\Psi^-\rangle); F_{\max}(\rho_{wer}) = \frac{1+p}{2} \qquad (3)$$

Quantum Advantage: $\rightarrow p > 1/3$

[2] Phys. Lett. A 222, 21 (1996).

Repeaters and Entanglement Swapping

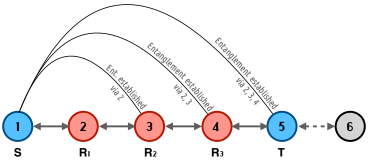


Figure: The Process of Entanglement Swapping

Starting Point

- Nodes are parties and links being **bipartite Werner states**.
- We are given a ground to ground repeater based network, where every node act as a **source and target** for teleportation
- We are given a ground to ground repeater based network, where each node can act as a **repeater station** in establishing a link between a source and target.
- There are **ensembles of entangled states** between two nodes: As with each swapping entanglement is destroyed

Repeaters and Entanglement Swapping

Step-1: First Swapping (Iteration)
Let ρ_{12}^1 be a Werner state between Alice and Bob and similarly ρ_{23}^1 be a Werner state between Bob and Charlie. These states are given by,

$$\rho_{12}^1 = p\Psi^-\Psi^- + \frac{1-p}{4}I,$$
$$\rho_{23}^1 = p\Psi^-\Psi^- + \frac{1-p}{4}I. \tag{4}$$

The teleportation fidelity of each of these states will be: $F_{\max}(\rho) = \frac{1+p}{2}$
After swapping the output will be [3]

$$\rho_{13}^2 \qquad = p^2\Psi^-\Psi^- + \frac{1-p^2}{4}I \tag{5}$$

and the teleportation fidelity of ρ_{13} as $\frac{1+p^2}{2}$.

[3]There can be multiple options. Result is not going to change for other options

Repeaters and Entanglement Swapping

Step-2: Second Swapping (Iteration)

$$\rho_{14}^{3} = p^{3}\Psi^{-}\Psi^{-} + \frac{1-p^{3}}{4} I \tag{6}$$

and the teleportation fidelity of ρ_{14} as $\frac{1+p^3}{2}$.

Step-k: k-th Swapping (Iteration)

$$\rho_{1,k+2}^{k+1} = p^{k+1}\Psi^{-}\Psi^{-} + \frac{1-p^{k+1}}{4} I \tag{7}$$

and the teleportation fidelity of $\rho_{1(k+2)}$ as $\frac{1+p^{k+1}}{2}$.

Note: If $p_i \neq p$, each one of the parameters are different , then **after Entanglement Swapping** the teleportation fidelity of the link connecting the source (S) and the target (T) can be calculated analytically as $F_{\mathrm{ST},\mathcal{P}}^{\max}(\rho_{\mathrm{wer}}) = \left(1 + \prod_{i\in\mathcal{P}} p_i\right)/2$

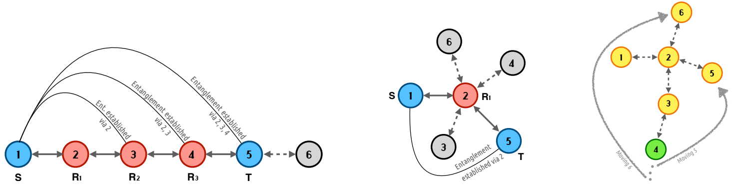
Average of Maximum Teleportation Fidelity

In a network through a particular path ($\mathcal{P}$) (having k intermediate links each of weigh p) if we establish a link between a particular source (S) and target (T) after swapping, then the teleportation fidelity of final link is given by $\frac{1+p^k}{2}$.
If S and T are connected via multiple paths (e.g., in networks with loops), $\mathcal{P}_{\max}$ be the path(s) with the **maximum fidelity**. we take the path with maximum fidelity. We get the average highest-achievable teleportation fidelity if we take the average of $F^{\max}_{\mathrm{ST},\mathcal{P}_{\max}}(\rho_{\mathrm{wer}})$ over all possible combinations of S and T :

Average of Maximum Teleportation Fidelity

$$F^{\max}_{\mathrm{avg}}(\rho_{\mathrm{wer}}) = \langle F^{\max}_{\mathrm{ST},\mathcal{P}_{\max}}(\rho_{\mathrm{wer}})\rangle_{\mathrm{ST}}$$
$$= \langle F^{\max}_{\mathcal{P}}(\rho_{\mathrm{wer}})\rangle_{\mathcal{P}} \quad (\mathrm{loopless}), \tag{8}$$

Average of Maximum Teleportation Fidelity



Three Loop less Topologies

Quantum repeater networks with $N = 6$ nodes (stations) and $L = 5$ links (shared states): the chain (left), the star (middle), and the second intermediate flower (right). To establish entanglement between Node 1 (source) and Node 5 (target) (shown in blue), the intermediate repeater stations (R_i, in red) perform entanglement swappings. Intermediate flowers are some specific tree networks. .

Average of Maximum Teleportation Fidelity

- **SCENARIO A:** All Werner state no Maximally Entangled State (all $p_i = p$ where $0 \leq p < 1$)
- **SCENARIO B:** M maximally entangled state ($p_i \in \{p, 1\}$, for M ME and all the others have $p_i = p$)
- **SCENARIO C:** the p_i's are randomly sampled from the uniform distribution, $\mathcal{U}_{[0,1]}$

Networks with out Loops

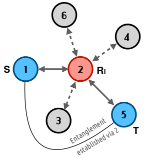
STAR NETWORK: A quantum star network of N nodes and $L = N - 1$ links. All links have the same weight p.

Scenario A: Star Network

$$F_{\text{avg}}^{\max}(N,p)\Big|_{\text{star}} = \left({}^{L}C_1\mathcal{F}_1 + {}^{L}C_2\mathcal{F}_2\right)/({}^{N}C_2), \tag{9}$$

where $\mathcal{F}_n \equiv (1 + p^n)/2$.

Intuition:



Out of the ${}^{N}C_2$ possible paths, $N - 1$ have length one and the rest ${}^{(N-1)}C_2$ have length two.

Networks with out Loops

- Possibilities of selecting two nodes out of N nodes $= {}^{N}C_2$.
- Total number of possibilities of selecting one path lengths $= {}^{N-1}C_1$ as there are (N-1) nodes connected to central node and selecting one node out of (N-1) nodes can be done in ${}^{N-1}C_1$ ways. Teleportation Fidelity contribution : $= {}^{N-1}C_1(\frac{1+p}{2})$.
- Possibilities of selecting two path lengths $= {}^{N-1}C_2$ as there are (N-1) nodes connected to central node and selecting two nodes out of (N-1) nodes passing through central node can be done in ${}^{N-1}C_2$ ways. Teleportation fidelity contribution: $= {}^{N-1}C_2(\frac{1+p^2}{2})$

Networks with out Loops

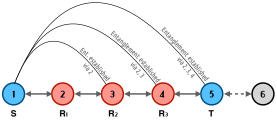
CHAIN NETWORK: A quantum chain network of N nodes and $L = N - 1$ links. All links have the same weight p.

Scenario A: Chain Network

$$F_{\rm avg}^{\rm max}(N,p)\big|_{\rm chain} = \frac{1}{{}^{N}C_2}\left[\sum_{\ell=1}^{L}(N-\ell)\mathcal{F}_\ell\right]. \tag{10}$$

where $\mathcal{F}_n \equiv (1+p^n)/2$.

Intuition:



Out of the ${}^{N}C_2$ possible paths, $N-1$ have length one, $N-2$ of length 2 and $N-l$ have length l.

Networks with out Loops

Chain of N nodes and $N-1$ edges:

Different path lengths	Number	Teleportation Capacity
1	$(N-1)$	$(N-1)\frac{1+p}{2}$
2	$(N-2)$	$(N-2)\frac{1+p^2}{2}$
$\cdots$	$\cdots$	$\cdots$
$N-1$	1	$\frac{1+p^{N-1}}{2}$

The total average fidelity for the linear chain without maximally entangled states is given by,

$$F_{avg}^{tell} = \frac{1}{{}^{N}C_2}\left[\sum_{l=1}^{N-1}(N-l)\frac{(1+p^l)}{2}\right]. \quad (11)$$

Networks with out Loops

FLOWER NETWORK: A quantum flower network of N nodes and $L = N - 1$ links. All links have the same weight p. The k-th IF is obtained after transferring k nodes from chain to star.
It can be viewed as a star of $k+3$ nodes ($k+2$ links) plus a chain of $N-k-2$ nodes ($L-k-2$ links).

Scenario A: Flower Network

$$\begin{aligned} F_{\text{avg}}^{\max}(N,p)\Big|_{\text{flower}_k} \\ = \frac{1}{{}^{N}C_2}\Bigg[\left\{{}^{k+2}C_1\mathcal{F}_1 + {}^{k+2}C_2\mathcal{F}_2\right\} + \left\{\sum_{\ell=1}^{L-k-2}(N-k-2-\ell)\mathcal{F}_\ell\right\} \\ + \left\{\sum_{\ell=1}^{L-k-2}\big((k+1)\mathcal{F}_{(\ell+2)} + \mathcal{F}_{(\ell+1)}\big)\right\}\Bigg] \end{aligned} \tag{12}$$

Networks with out Loops

Intuition: The first and second sets of terms (numerator) come from the star and the chain, respectively, and the third set comes from overlapping paths connecting these two structures.

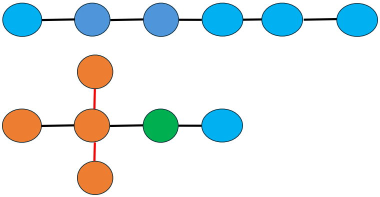


Figure: Formation of Branching Tree with $N = 6$ and $k = 2$

An Example :

For $N = 6$, $k = 2$, we obtain star of 5 nodes, and a chain of 2 nodes. Star contributes upto path length 2 and chain 1 ($l = 1$). The overlapping paths contribute up to path length 3 ($l + 2$)

Networks with out Loops

STAR NETWORK: A quantum star network of N nodes and $L = N - 1$ links. M out of the L links are maximally entangled.

Scenario B: Star Network

$$F_{\mathrm{avg}}^{\mathrm{max}}(N, M, p)\Big|_{\mathrm{star}} = \frac{1}{{}^{N}C_2}\left[{}^{M+1}C_2\mathcal{F}_0 + (M+1)(L-M)\mathcal{F}_1 + {}^{L-M}C_2\mathcal{F}_2\right]. \quad (13)$$

where $\mathcal{F}_n \equiv (1 + p^n)/2$.

Intuition: 'M' maximally entangled states and '$N - M - 1$' without maximally entangled states : For one path length $\rightarrow$ 1. the path is maximally entangled or 2. the path is without maximally entangled. Teleportation Fidelity for one path length $=$ $M(\frac{1+p^0}{2}) + (N - M - 1)(\frac{1+p^1}{2})$.

Networks with out Loops

Intuition : Now, considering two path lengths in Star Network, there are three possibilities.

- **Possibility 1 (Both edges are maximally entangled):** The possibility of selecting two out of 'M' maximally entangled states = ${}^{M}C_2$ and teleportation fidelity = ${}^{M}C_2(\frac{1+p^0}{2})$
- **Possibility 2 (Both edges are without maximally entangled state):** The possibility of selecting two out of 'N-M-1' without maximally entangled states = ${}^{N-M-1}C_2$ and teleportation fidelity = ${}^{N-M-1}C_2(\frac{1+p^2}{2})$
- **Possibility 3 (One of the edge is maximally entangled):** The possibility of selecting two links out of which one is maximally entangled state is $({}^{N-1}C_2 - {}^{M}C_2 - {}^{N-M-1}C_2)$ and teleportation fidelity = $({}^{N-1}C_2 - {}^{M}C_2 - {}^{N-M-1}C_2)(\frac{1+p^1}{2})$

Network without Loops

Take Away From Scenario B:

Role of a maximally entangled state:

- For Star network the number of maximally entangled states only matters
- For Chain and Flower network both position and number of maximally entangled state will matter

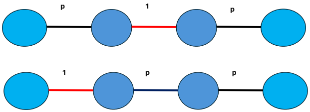


Figure: Two different locations of ME

For p=1/2 : In the first case the average fidelity is 0.77 and the second case it is 0.75.

Networks with out Loops

Take Away From Scenario B:
The Quantum Advantage for a Link : If the link is Werner state,

$$\rho_{\text{wer}} = \frac{1-p}{4}\mathbb{I} + pP(|\Psi^-\rangle); F_{\max}(\rho_{wer}) = \frac{1+p}{2} \tag{14}$$

Quantum Advantage for the link : When $\rightarrow F_{\max}(\rho_{wer}) > 2/3, p > 1/3$.

The Quantum Advantage for Network:

- A large quantum network as a whole is expected to show quantum advantage if $F_{\text{avg}}^{\max} > 2/3$.
- This is because, without entangled states, each path can only achieve a maximum teleportation fidelity of 2/3. Hence, at that threshold, the average maximum fidelity also becomes 2/3. [Our interpretation]
- Other interpretations: 1) At least one path in the network shows quantum advantage 2) All paths in the network show quantum advantages

Networks with out Loops

Take Away From Scenario B:

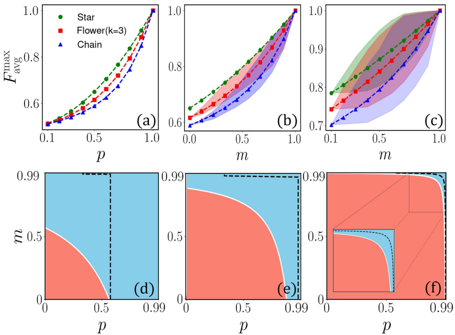


Figure: (Top panel $N = 10$) The dependence of $F_{\rm avg}^{\rm max}$ on p and $m = M/L$. (Bottom panel $N = 100$) Regime of Quantum Advantage

- Dependence on p and $m = M/L$ for $N = 10$ (top row)
- Since the position of ME plays a role, so we will get a band of fidelities for Chain and flower.
- Dots are numerically simulated results. Lines are from theory.
- Plot(c) for scenario:C
- The lower row plots are obtained by averaging over all possible arrangements of the ME links (N=100).
- The right of dashed line all path has a teleportation fidelity $> 2/3$

Networks with out Loops

Path Length Implications

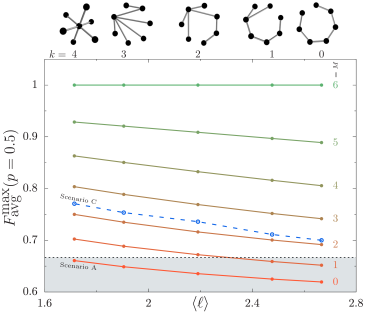
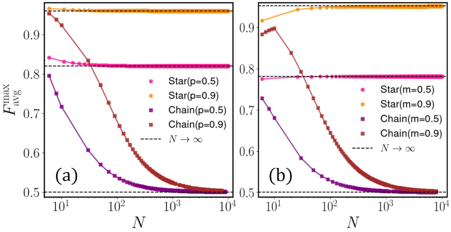


Figure: The average teleportation fidelity, $F_{\rm avg}^{\rm max}$,(p=1/2) of seven-node networks; $F_{\rm avg}^{\rm max} < 2/3$ in the shaded region. Bottom red line: Scenario A, Other solid lines: Scenario B and Blue dashed line : Scenario C.

- As expected, $F_{\rm avg}^{\rm max}$ decreases as we go from the star to the chain.
- The situation improves with the introduction of ME links. For $2 \leq M \leq 6$, all graphs can achieve **quantum advantage** for the same value of p.
- for $M = 1$, only the second or higher intermediate flowers show $F_{\rm avg}^{\rm max} > 2/3$

Networks with out Loops

Large N Behaviour:



Implications

$F_{avg}^{max} > 1/2$ for $m < 1$

- Two bench mark choices for star for varying p, ($p = 0.5$, $p = 0.9$)
- Two bench mark choices for star for varying m, ($m = 0.5$, $m = 0.9$)
- Two bench mark choices for chain for varying p, ($p = 0.5$, $p = 0.9$)
- Two bench mark choices for chain for varying m, ($m = 0.5$, $m = 0.9$)

Networks with Loops

RING AND COMPLETE NETWORK :

- In Network with loops there can be in principle more than 1 path between a source and target.
- We should look for the best possible path as per our definition.
- For uniform p, the shortest path is the best path. However this is not the case in case of non uniform p.

Two Examples:

- A simple ring : Chain with two ends connected
- Complete Graph: There is a direct link between each pair of nodes

Networks with Loops

RING AND COMPLETE NETWORK :

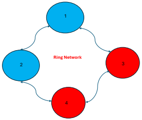


Figure: A ring network with $N = 4$

Ring Network: Scenario A

In a ring there are always two paths between any two nodes S and T of a ring. Selecting the shorter path we obtain

$$F_{\mathrm{avg}}^{\mathrm{max}}(N,p)\big|_{\mathrm{ring}} = \frac{1}{\lfloor N/2 \rfloor}\left[\sum_{\ell=1}^{\lfloor N/2 \rfloor} \mathcal{F}_\ell\right]. \qquad (15)$$

Note, when N is even , a pair of opposite nodes are connected to two equal length both contributing equally. Here $\lfloor . \rfloor$ is the floor function.

Networks with Loops

RING AND COMPLETE NETWORK :

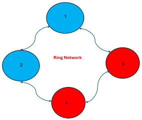


Figure: A ring network with $N = 4$

- Different paths in the above network for different source and targets (example):
  $(S, T) : (1,2) \rightarrow \mathbf{(1,2)}, (1,4,3,2); (1,3) \rightarrow (1,2,3)(1,4,3); (1,4) \rightarrow (1,4); (2,3) \rightarrow (2,3); (2,4) \rightarrow (2,3,4)(2,1,4); (3,4) \rightarrow (3,4)$
- 4 paths of length 1 and 4 paths of length 2
- For each (S, T) if there is a k length path, then there is $N - k$ length path.
- For $N = 4$, the average of maximum teleportation fidelity comes out to be $66/96$

Networks with Loops

RING AND COMPLETE NETWORK :

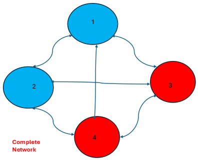


Figure: A complete graph with $N = 4$

Complete Network: Scenario A

It is easy to figure out that the shortest path achieve the maximum fidelity as $p^2 < p$. Every S and T contributes $\frac{1+p}{2}$.

$$F_{\text{avg}}^{\max}(N, p)\Big|_{\text{complete}} = \frac{1+p}{2}. \tag{16}$$

Networks with Loops

RING AND COMPLETE NETWORK :

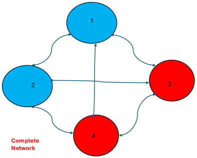


Figure: A complete graph with $N = 4$

- Shortest (Best Fidelity) paths in the above network for different source and targets: $(S, T) : (1, 2) \rightarrow (1, 2)$
- For $p = 0.5$ the average of maximum teleportation fidelity turns out to be $72/96(0.75)$.

Impact and Conclusions

Total Take away:

- We can define a meaningful global metric across a quantum network.
- Teleportation ability across a quantum internet could be modeled as easily as counting path lengths in a graph?
- The quantum advantage of a network for distributed teleportation is different from quantum advantage of a state.
- Helps in ranking the best network for teleportation for a given N and p.

Impact and Conclusions

CONCLUSIONS:

- Platform: Quantum Network
- Technology : Repeater Based Technology
- Motivation: Find a global measure to characterize a network's capability to carry out distributed Teleportation F_{avg}^{max} (Measure of resourcefulness of a network as a whole).
- Model: Links of the Networks are Werner state
- Scenarios: (1) all $p_i = p$, $0 < p < 1$. (2) A fraction of links are ME 3) The p_i's are randomly sampled from uniform distribution.
- Comparison: Across topologies $\rightarrow$ Loop less and Loops (Obtaining Analytical closed expression for the measure)

Thank You

My Collaborators :

- Subhadip Mitra
- Chittaranjan Hens
- Mylavarappu Ganesh
- Subrata Ghosh

Thank You

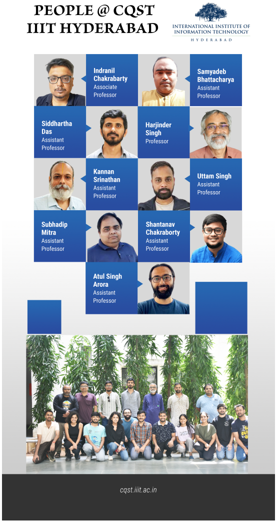


Figure: OUR CENTRE

Thank You

END