Relative Nature of Quantum Complementarity: A No-Comparison Theorem

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Complementarity is a conceptual aspect of quantum mechanics that Niels Bohr regarded as an essential feature of the theory. The complementarity principle holds that experimental arrangements enabling unambiguous determination of certain physical properties are mutually exclusive.

Naive idea of complementarity:

Two ideal observables E_1 *and* E_2 are complementary when one of them is property of the system, then the other cannot be. Mathematically:

$$p_T^{E_1}(X) = Tr[TE_1(X)] = 1$$

$$p_T^{E_2}(Y) = Tr[TE_2(Y)] < 1$$

$for\ all\ X\ and\ Y\ and\ density\ matrix\ T\ and\ E_1(X), E_2(X) \neq 0, I$

Example:

a) Wave-particle property in double slit experiment.

b) Position and momentum of a particle.

$$\langle\psi|E_Q(X)|\psi\rangle = 1 \Rightarrow \langle\psi|E_P(Y)|\psi\rangle < 1$$

c) Spin property along different directions.

$$p(\sigma_z = 1|\psi) = 1 \Rightarrow p(\sigma_x = \pm 1|\psi) < 1$$

Joint Measurement:

Two observable $\boldsymbol{E_1(X)}$ *and* $\boldsymbol{E_2(Y)}$ are jointly measurable if there exist a POVM $\boldsymbol{M(X \times Y)}$ on $\mathbb{R}^2$ such that

$$\left.\begin{aligned} \boldsymbol{E_1(X)} &= \boldsymbol{M(X \times \mathbb{R})} \\ \boldsymbol{E_2(X)} &= \boldsymbol{M(\mathbb{R} \times Y)} \end{aligned}\right\} \textit{ for all } \boldsymbol{X} \textit{ and } \boldsymbol{Y}$$

Measurement-theoretical characterisation of complementarity:

Two observables $\boldsymbol{E_1}$ *and* $\boldsymbol{E_2}$ are complementary if and only if their associated measurement arrangement (Instrument) are mutually exclusive.

Mathematically:

$$\boldsymbol{I^{E_1}(X) \wedge I^{E_2}(Y) = 0}$$

$$\Rightarrow \boldsymbol{E_1(X) \wedge E_2(Y) = 0}$$

Position ($\boldsymbol{Q}$) and Momentum ($\boldsymbol{P}$):

$$\boxed{\begin{aligned} &\boldsymbol{E_Q(X) \wedge E_P(Y) = 0} \\ &\boldsymbol{E_Q(\mathbb{R}/X) \wedge E_P(Y) = 0} \\ &\boldsymbol{E_Q(X) \wedge E_P(\mathbb{R}/Y) = 0} \end{aligned}}$$

Possibility of joint measurement of position and momentum by introducing unsharp counterpart (POVM):

$$E_Q^f(X) = \int f(x)\, E_Q(x+X)\, dx, \qquad f(x) \geq 0, \int f dx = 1$$

$$E_P^g(Y) = \int f(y)\, E_P(y+Y)\, dy, \qquad g(y) \geq 0, \int g dy = 1$$

$$Var\left(E_Q^f, \psi\right) = Var\,(E_Q, \psi) + Var(f)$$

$$Var(E_P^g, \psi) = Var\,(E_P, \psi) + Var(g)$$

Choose the following POVM:

$$E(X \times Y) = \int_{X\times Y} U_{qp} |\phi\rangle\langle\phi| U_{qp}^{\dagger} dp dq \quad \text{where} \quad U_{qp} = e^{i(p\hat{Q} - q\hat{P})}$$

$$E(X \times \mathbb{R}) = E_Q^f(X), \qquad E(\mathbb{R} \times Y) = E_P^g(Y)$$

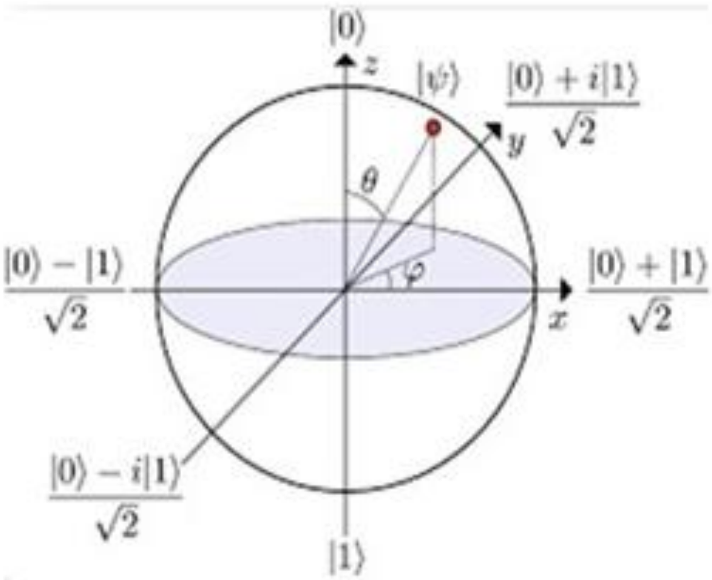
$$\textit{Choose} \;\; f = |\phi|^2 \qquad g = |\bar{\phi}|^2$$

$$Var(f)Var(g) \geq \frac{\hbar^2}{4}$$

$$\Delta Q \Delta P \geq \frac{\hbar}{2}$$

ΔQ and ΔP being error in position and momentum measurement respectively.

Come to Spin System:



Spin State:

$|\psi\rangle\langle\psi| = \frac{1}{2}[I + \boldsymbol{m}.\boldsymbol{\sigma}]$, $\boldsymbol{m}$ *being an unit vector.*

Spin Observable:

$$\sigma_n = \boldsymbol{n}.\boldsymbol{\sigma} = (+1)\,P_n^+ + (-1)P_n^-$$

$P_n^{\pm} = \frac{1}{2}[I \pm \boldsymbol{n}.\boldsymbol{\sigma}]$, $\boldsymbol{n}$ *is a unit vector*

Born probability Rule:

$$p(\sigma_n = \pm 1|\psi) = Tr\left[|\psi\rangle\langle\psi|P_n^{\pm}\right] = \frac{1}{2}(1 + \boldsymbol{n}.\boldsymbol{m})$$

Perfect spin measurement along different directions are complementary as

$$P_{n_1}^{\pm} \wedge P_{n_2}^{\pm} = 0\,, \forall\ n_1 \neq n_2$$

Unsharp Soin obsevable:

$$\sigma_n(\lambda) = E_n^a(\lambda) = \left\{\frac{1}{2}[I + \lambda\, a\, \boldsymbol{n}.\boldsymbol{\sigma}], a = \pm 1,\ 0 \leq \lambda \leq 1\right\}$$

$$E_n^{\pm}(\lambda) = \frac{1+\lambda}{2}\ \frac{1}{2}[I \pm \boldsymbol{n}.\boldsymbol{\sigma}] + \frac{1-\lambda}{2}\ \frac{1}{2}[I \mp \boldsymbol{n}.\boldsymbol{\sigma}]$$

$$\boxed{E_n^a(\lambda) = \lambda P_n^a + (1-\lambda)\frac{I}{2}}$$

λ *being the degree of unsharpness.*

Condition for joint spin measurement:

$\{\sigma_{n_i}(\lambda)\}_{i=1}^{k}$ is jointly measurable if there exists a

POVM: $G = \{\Pi_{\vec{a}} \geq 0, \sum_{\vec{a}} \Pi_{\vec{a}} = I\}$ such that

$$E_{n_r}^{a_r}(\lambda) = \sum_{\vec{a}/a_r} \Pi_{\vec{a}} \quad \forall\, r \text{ and } a_r$$

$$\vec{a} = [a_1 a_2 \cdots a_k] = \in [+1, -1]^k$$

Condition of joint measurement of unsharp spin $\sigma_n(\lambda)$ *and* $\sigma_m(\lambda)$ along two arbitrary directions:

Four outcome POVM: $\{\Pi_{++}, \Pi_{+-}, \Pi_{-+}, \Pi_{--}\}$, **such that**

$$E_n^+(\lambda) = \Pi_{++} + \Pi_{+-}, \qquad E_n^-(\lambda) = \Pi_{-+} + \Pi_{--}$$

$$E_m^+(\lambda) = \Pi_{++} + \Pi_{-+}, \qquad E_m^-(\lambda) = \Pi_{-+} + \Pi_{--}$$

$$\Rightarrow$$

$$\lambda(|n + m| + |n - m|) \leq 2$$

$$Max(|n + m| + |n - m|) = 2\sqrt{2}$$

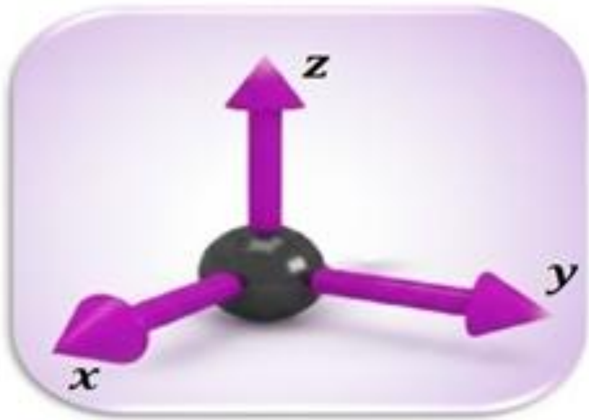
$$\lambda_{opt} \leq \frac{1}{\sqrt{2}}$$

(P. Busch, 1986 and M. Banik et al, 2013)

For a set observables Degree of complementarity is captured by the value of λ.

Spin observable along x and y directions are compatible as long as

$$\lambda \leq \frac{1}{\sqrt{2}} = \lambda_{Opt}$$

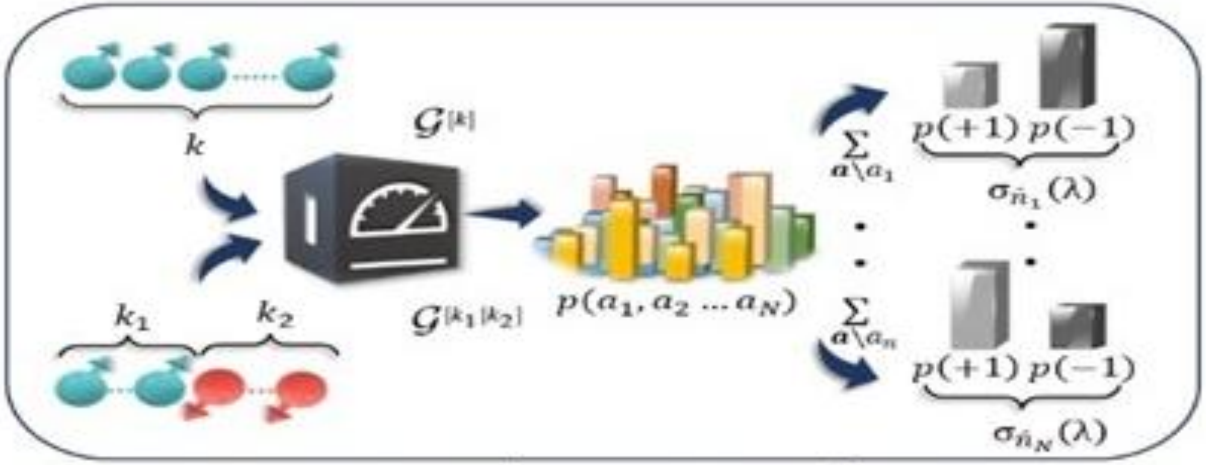


Spin observable along x, y and z directions are compatible as long as

$$\lambda \leq \frac{1}{\sqrt{3}} = \lambda_{Opt}$$

A New paradigm regarding complementarity:

(Possibility of reducing complementarity by increasing copies)



The set of spin observables $(\sigma_1, \sigma_2, \dots \sigma_n)$ is said to be k-copy jointly measurable if there exists a POVM:

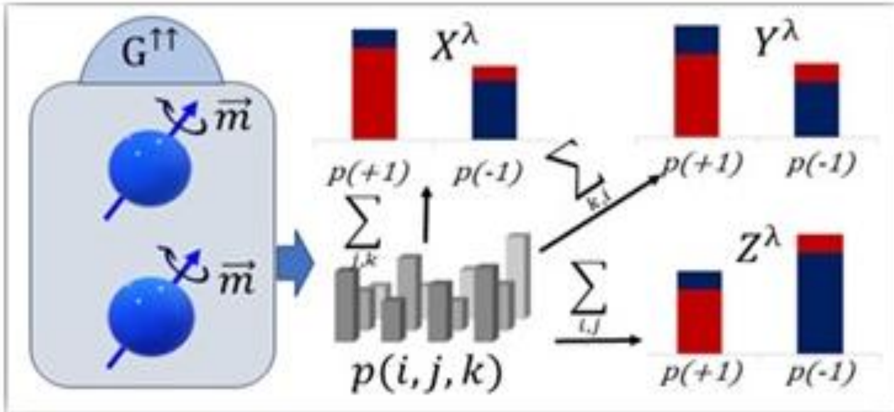
$$G = \left\{ \begin{matrix} \Pi_{\vec{a}} \in \mathcal{L}\left(\left((\mathbb{C}^2)^{\otimes k}\right) \middle| \Pi_{\vec{a}} \geq 0, a \in \{+,-\}^k \right. \\ and \sum_{\vec{a}} \Pi_{\vec{a}} = I^{\otimes k} \end{matrix} \right\}$$

On k *copies* of the system, such that for all states ρ_m

$$Tr\left[\rho_m \mathrm{E}_{\mathrm{j}}^{\mathrm{a_j}}(\lambda)\right] = \sum_{\vec{a}/a_j} \mathrm{Tr}[\rho_m^{\otimes k}\, \Pi_{\vec{a}}]$$

and for all $j \in \{1, 2 \dots n\}$. (Carmeli et al, 2016)

$$\{\sigma_x(\lambda), \sigma_y(\lambda), \sigma_z(\lambda)\}$$



$$G = \{\Pi^{\uparrow\uparrow}_{\vec{a}}\}_{\vec{a}} \in \ (\mathbb{C}^2)^{\otimes 2}, \Pi^{\uparrow\uparrow}_{\vec{a}} \geq 0, \sum \Pi^{\uparrow\uparrow}_{\vec{a}} = I^{\otimes 2},\ \vec{a} \in [+,-]^3$$

$$Tr[\rho_n E^{+}_{\alpha}(\lambda)] = \sum_{\vec{a}/a_r} Tr[\Pi^{\uparrow\uparrow}_{a} \rho_n{}^{\otimes 2}]\ \forall \rho_n\ and\ E^{\pm}_{\alpha}(\lambda), \alpha = x, y, z$$

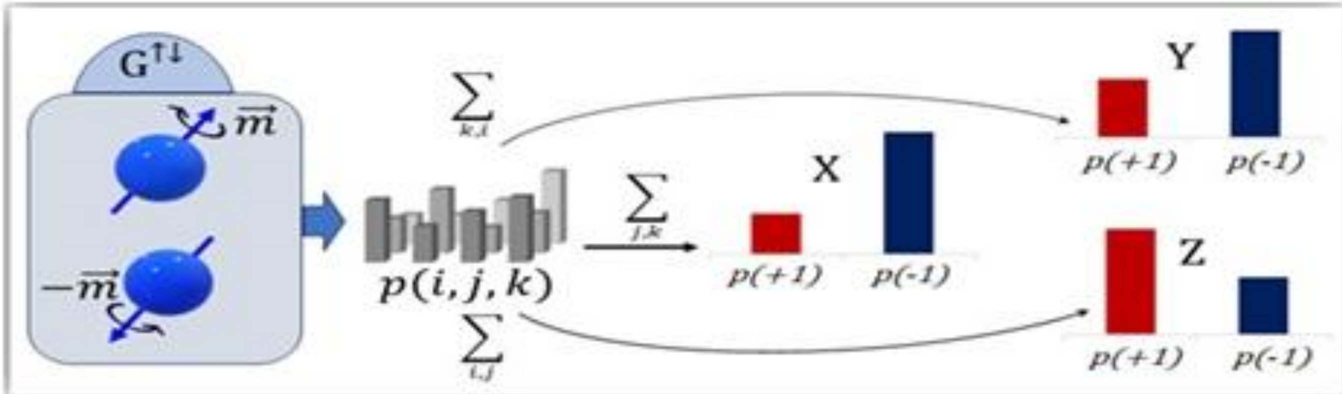
$$Tr[\rho_n E^{+}_{x}] = Tr[\left(\Pi^{\uparrow\uparrow}_{+++} + \Pi^{\uparrow\uparrow}_{++-} + \Pi^{\uparrow\uparrow}_{+-+} + \Pi^{\uparrow\uparrow}_{+--}\right)\rho_n{}^{\otimes 2}]$$

$$\Pi^{\uparrow\uparrow}_{ijk} = \frac{1}{32}\left[4I^{\otimes 2} + \sqrt{3}\left(i\{\{X,I\}\} + j\{\{Y,I\}\} + k\{\{Z,I\}\}\right) + ij\{\{X,Y\}\} + jk\{\{Y,Z\}\} + ki\{\{Z,X\}\}\right]$$

Where $\{\{A,B\}\} = A \otimes B + B \otimes A$

$$\lambda \leq \frac{\sqrt{3}}{2} > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}}$$

Anti-parallel case:



Theorem 1. *The observables X^{λ}, Y^{λ}, and Z^{λ} are jointly measurable on antiparallel spin pairs for all $\lambda \in [0,1]$.*

(Patra et al, 2025)

$$\Pi_{ijk}^{\uparrow\downarrow} = \frac{1}{16}\Big(2I^{\otimes 2} + i\big[[X,I]\big] + j\big[[Y,I]\big] + k\big[[Z,I]\big] - ij\big\{\{X,Y\}\big\} - jk\big\{\{Y,Z\}\big\} - ki\big\{\{Z,X\}\big\}\Big)$$

where $\big[[U,V]\big] = U \otimes V - V \times U$

$$\lambda_{opt}^{\uparrow\downarrow} = 1 \Rightarrow \lambda_{opt}^{\uparrow\downarrow} > \lambda_{opt}^{\uparrow\uparrow} = \frac{\sqrt{3}}{2}$$

The Relativity of Quantum Complementarity: A No-Comparison Theorem

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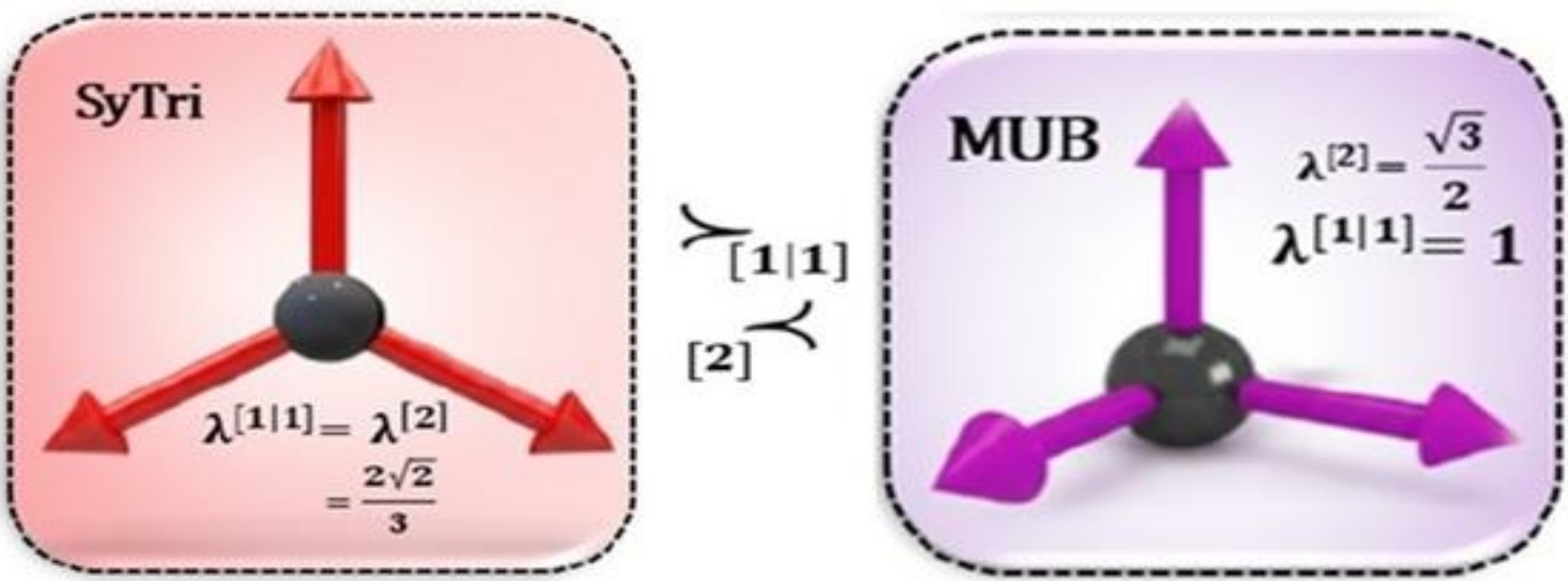
We establish a No-Comparison Theorem showing that the relative complementarity of two sets of observables can be reversed simply by changing the configuration in which the systems are prepared. This foundational insight, on the practical front, compels a fundamental rethinking of quantum information protocols under finite-resource constraints.

Convention:

Smaller the value of λ for a set of observables, more is the complementarity of the set.

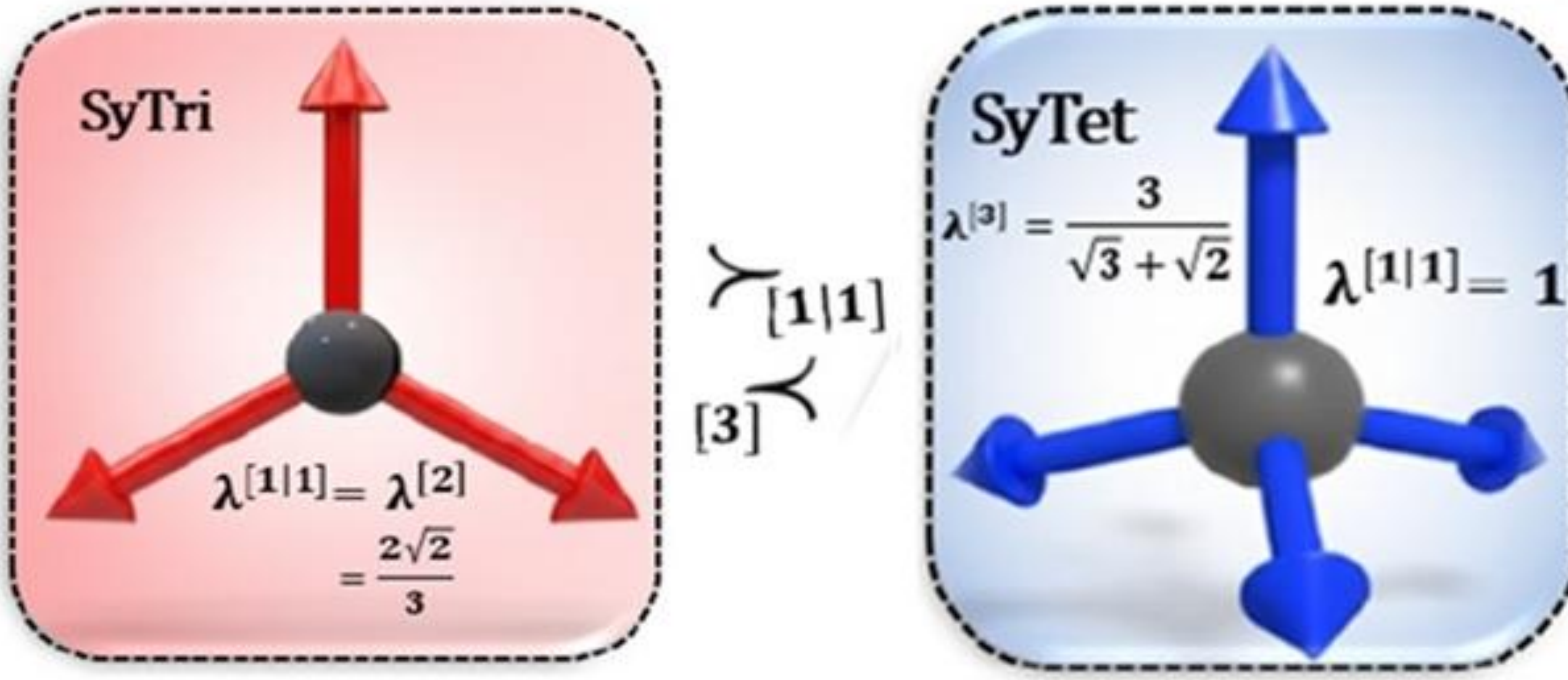
No-Comparison Theorem

Example-1



No-Comparison Theorem

Example-2



Conclusions:

- ✓ we explore a pivotal, yet overlooked, question: How does the quantum limitation of joint measurability vary when multiple copies of a quantum system are available? How does it depend on the configuration of the finite state ensemble?

- ✓ We demonstrate that the landscape of complementarity undergoes a radical shift in this multi-copy regime. The very ordering of sets of observables in terms of their degree of complementarity, once thought to be an intrinsic property of the observables themselves, becomes profoundly configuration-dependent.

- ✓ The observed configuration dependence of quantum complementarity originates from entanglement induced in the measurement interaction, rather than from entanglement among the states themselves, underscoring the active role of measurement dynamics in determining quantum incompatibility.

- ✓ This foundational insight, on the practical front, compels a fundamental rethinking of quantum information protocols under finite-resource constraints.