Quantification of multipartite entanglement as a resource and the role of measurement processes for multiqubit systems

Debasis Sarkar

Department of Applied Mathematics, University of Calcutta, 92, A.P.C. Road, Kolkata- 700009, India

ICQIST, December 9 - 13, 2025

This lecture is based on two of our recent works:
Ref: 1. Entanglement of assistance as a measure of multiparty entanglement, Phys. Rev. A, 111 (2025) 032423 and

2. Increasing entanglement concentration via measurement-imaginarity, communicated.

Collaborators: Indranil Biswas, Subrata Bera, Atanu Bhunia, Indrani Chattopadhyay and Ujjwal Sen"

Introduction

- The nature of multipartite entanglement is quite different from that of bipartite entanglement

Introduction

- The nature of multipartite entanglement is quite different from that of bipartite entanglement
- It is really a difficult to generalize the concepts of bipartite entanglement to multipartite cases

Introduction

- The nature of multipartite entanglement is quite different from that of bipartite entanglement
- It is really a difficult to generalize the concepts of bipartite entanglement to multipartite cases
- The question of classification of multipartite entanglement as well as the quantification procedures are also not quite easy to handle

Introduction

- The nature of multipartite entanglement is quite different from that of bipartite entanglement
- It is really a difficult to generalize the concepts of bipartite entanglement to multipartite cases
- The question of classification of multipartite entanglement as well as the quantification procedures are also not quite easy to handle
- Several questions we are facing whenever we deal with multipartite quantum systems

Introduction

- The nature of multipartite entanglement is quite different from that of bipartite entanglement
- It is really a difficult to generalize the concepts of bipartite entanglement to multipartite cases
- The question of classification of multipartite entanglement as well as the quantification procedures are also not quite easy to handle
- Several questions we are facing whenever we deal with multipartite quantum systems
- For example:- what do we mean by a maximally entangled state in a multiqubit system?

3-qubit system

Consider the three qubit system:

• The 1st figure on the left denote the fully separable three-qubit pure state $|\psi\rangle = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C$.

• The 2nd figure from the left denote the biseparable three-qubit pure state in the partition $A|BC$: $|\psi\rangle = |\alpha\rangle_A \otimes |\phi\rangle_{BC}$, where $|\phi\rangle$ may be an entangled state.

Figure: Fully separable and biseparable states

3-qubit system

Consider the three qubit system:

• The 1st figure on the left denote the fully separable three-qubit pure state $|\psi\rangle = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C$.

• The 2nd figure from the left denote the biseparable three-qubit pure state in the partition $A|BC$: $|\psi\rangle = |\alpha\rangle_A \otimes |\phi\rangle_{BC}$, where $|\phi\rangle$ may be an entangled state.

Figure: Fully separable and biseparable states

• The 3rd and 4th figures from the left denote the biseparable three-qubit state in the partitions $C|AB$ and $B|AC$ respectively: $|\psi\rangle = |\alpha\rangle_C \otimes |\phi\rangle_{AB}$ and $|\psi\rangle = |\alpha\rangle_B \otimes |\phi\rangle_{AC}$, where $|\phi\rangle$ may be an entangled state.

Contd..

- Several types of genuine entanglement can exist in three-qubit states depending on the presence of entanglement in the reduced party systems.

Contd..

- Several types of genuine entanglement can exist in three-qubit states depending on the presence of entanglement in the reduced party systems.
- The 1st figure in the left denotes those states that become separable upon reducing any of the three parties. e.g.: $|\psi\rangle = \lambda_0 |000\rangle + \lambda_4 |111\rangle$

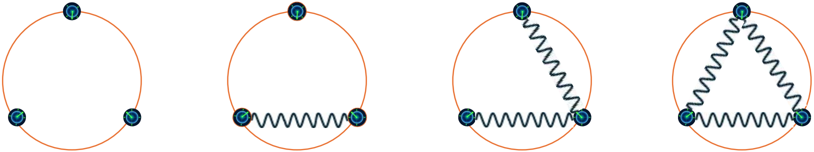


Figure: Reduced party entanglement structures of three-qubit genuinely entangled states

Contd..

- Several types of genuine entanglement can exist in three-qubit states depending on the presence of entanglement in the reduced party systems.
- The 1st figure in the left denotes those states that become separable upon reducing any of the three parties. e.g.: $|\psi\rangle = \lambda_0 |000\rangle + \lambda_4 |111\rangle$

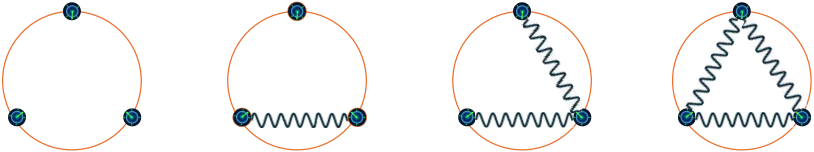


Figure: Reduced party entanglement structures of three-qubit genuinely entangled states

- The 2nd figure from the left denotes those states that remains non-separable upon reducing only one of the three parties.

Contd..

e.g.: $|\psi\rangle_{ABC} = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle$ such that $\mathcal{E}(\mathrm{Tr}_A |\psi\rangle\langle\psi|) \neq 0$ but $\mathcal{E}(\mathrm{Tr}_B |\psi\rangle\langle\psi|) = 0 = \mathcal{E}(\mathrm{Tr}_C |\psi\rangle\langle\psi|)$, with $\mathcal{E}$ being some measure of entanglement.

Contd..

e.g.: $|\psi\rangle_{ABC} = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle$ such that $\mathcal{E}(\mathrm{Tr}_A |\psi\rangle\langle\psi|) \neq 0$ but $\mathcal{E}(\mathrm{Tr}_B |\psi\rangle\langle\psi|) = 0 = \mathcal{E}(\mathrm{Tr}_C |\psi\rangle\langle\psi|)$, with $\mathcal{E}$ being some measure of entanglement.

- The 3rd figure from the left denotes those states that remains non-separable upon reducing two of the three parties. e.g.: $|\psi\rangle_{ABC} = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle$ such that $\mathcal{E}(\mathrm{Tr}_A |\psi\rangle\langle\psi|) \neq 0$, $\mathcal{E}(\mathrm{Tr}_B |\psi\rangle\langle\psi|) \neq 0$ but, $\mathcal{E}(\mathrm{Tr}_C |\psi\rangle\langle\psi|) = 0$.

Contd..

e.g.: $|\psi\rangle_{ABC} = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle$ such that $\mathcal{E}(\mathrm{Tr}_A |\psi\rangle\langle\psi|) \neq 0$ but $\mathcal{E}(\mathrm{Tr}_B |\psi\rangle\langle\psi|) = 0 = \mathcal{E}(\mathrm{Tr}_C |\psi\rangle\langle\psi|)$, with $\mathcal{E}$ being some measure of entanglement.

- The 3rd figure from the left denotes those states that remains non-separable upon reducing two of the three parties. e.g.: $|\psi\rangle_{ABC} = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle$ such that $\mathcal{E}(\mathrm{Tr}_A |\psi\rangle\langle\psi|) \neq 0$, $\mathcal{E}(\mathrm{Tr}_B |\psi\rangle\langle\psi|) \neq 0$ but, $\mathcal{E}(\mathrm{Tr}_C |\psi\rangle\langle\psi|) = 0$.
- The 4th figure from the left denotes those states that remains non-separable upon reducing all of the three parties. e.g.:

$|W\rangle = \frac{1}{\sqrt{3}} |100 + 010 + 001\rangle$,

$|\psi\rangle_{ABC} = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle$.

Thus we observe different kind of entanglement behaviour exists starting from three qubit systems.

Entanglement of Assistance

- Entanglement of Assistance for a three-qubit state is the maximal amount of help that C(one party) can provide locally to concentrate entanglement between A and B(other two parties).

[1] G. Gour, D. A. Meyer, and B. C. Sanders, *Phys. Rev. A 72, 042329 (2005)*.

Entanglement of Assistance

- Entanglement of Assistance for a three-qubit state is the maximal amount of help that C(one party) can provide locally to concentrate entanglement between A and B(other two parties).
- We use Concurrence of Assistance (CoA[1]) to form a new measure by considering the geometric mean of the CoAs with respect to three parties. We call it 'Volume of Assistance' (VoA).

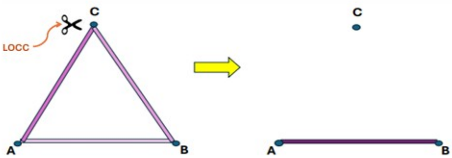


Figure: Localization of Entanglement

[1] G. Gour, D. A. Meyer, and B. C. Sanders, *Phys. Rev. A 72, 042329 (2005)*.

Contd..

- This measure is quite different from that of some of the existing measures.
- VoA forms an Upper bound of generalized geometric measure (GGM[2]) for specific classes of three qubit states.

[2] D. Sadhukhan, S. S. Roy, A. K. Pal, D. Rakshit, A. Sen(De), and U. Sen, *Phys. Rev. A 95, 022301 (2017)* and references therein.

[3] Dong-Dong Dong , Li-Juan Li, Xue-Ke Song, Liu Ye, and Dong Wang, *Phys. Rev. A 110, 032420 (2024)*.

Contd..

- This measure is quite different from that of some of the existing measures.
- VoA forms an Upper bound of generalized geometric measure (GGM[2]) for specific classes of three qubit states.
- A new measure 'Minimum Pairwise Concurrence'[3] (MPC) has been introduced recently. Remarkably, VoA can distinguish a broad class of states that evade differentiation by MPC.

[2] D. Sadhukhan, S. S. Roy, A. K. Pal, D. Rakshit, A. Sen(De), and U. Sen, *Phys. Rev. A 95, 022301 (2017)* and references therein.

[3] Dong-Dong Dong , Li-Juan Li, Xue-Ke Song, Liu Ye, and Dong Wang, *Phys. Rev. A 110, 032420 (2024)*.

Contd..

- This measure is quite different from that of some of the existing measures.
- VoA forms an Upper bound of generalized geometric measure (GGM[2]) for specific classes of three qubit states.
- A new measure 'Minimum Pairwise Concurrence'[3] (MPC) has been introduced recently. Remarkably, VoA can distinguish a broad class of states that evade differentiation by MPC.
- For any pure three-qubit state $|\psi\rangle_{ABC}$, its VoA is given by

$$\overline{\mathcal{C}}_3(|\psi\rangle) = \sqrt[3]{\mathcal{C}^a_A(|\psi\rangle) \cdot \mathcal{C}^a_B(|\psi\rangle) \cdot \mathcal{C}^a_C(|\psi\rangle)}$$

where $\mathcal{C}^a_A$, $\mathcal{C}^a_B$ and $\mathcal{C}^a_C$ denotes the CoA with respect to the corresponding local parties.

[2]D. Sadhukhan, S. S. Roy, A. K. Pal, D. Rakshit, A. Sen(De), and U. Sen, *Phys. Rev. A 95, 022301 (2017)* and references therein.

[3]Dong-Dong Dong , Li-Juan Li, Xue-Ke Song, Liu Ye, and Dong Wang, *Phys. Rev. A 110, 032420 (2024)*.

Contd..

- CoA for a tripartite state $|\psi\rangle_{ABC}$ is defined as

$$\mathcal{C}_C^a(|\psi\rangle_{ABC}) = max \sum_i p_i \mathcal{C}_{AB}(|\phi_i\rangle_{AB})$$

where $\{p_i, |\phi_i\rangle\}$ is a decomposition of ρ_{AB}.

Contd..

- CoA for a tripartite state $|\psi\rangle_{ABC}$ is defined as

$$\mathcal{C}_C^a(|\psi\rangle_{ABC}) = max \sum_i p_i \mathcal{C}_{AB}(|\phi_i\rangle_{AB})$$

where $\{p_i, |\phi_i\rangle\}$ is a decomposition of ρ_{AB}.
- Fortunately, for three-qubit systems, there is a closed form analytical expression of CoA defined by

$$\mathcal{C}_C^a(|\psi\rangle_{ABC}) = F(\rho_{AB}, \tilde{\rho_{AB}})$$

where $\tilde{\rho_{AB}} = \sigma_y \otimes \sigma_y \rho_{AB}^* \sigma_y \otimes \sigma_y$ (ρ_{AB}^* being the complex conjugate of ρ_{AB}) and $F(\rho_{AB}, \tilde{\rho_{AB}}) = \mathrm{Tr}\sqrt{\rho_{AB}^{1/2}\tilde{\rho_{AB}}\rho_{AB}^{1/2}}$ is the fidelity between ρ_{AB} and $\tilde{\rho_{AB}}$. Thus it makes CoA a computable, tripartite entanglement monotone for pure tripartite states.

Volume of Assistance Contd..

- VoA can be generalized for three-qudit states by considering generalized concurrence of assistance (GCoA) for certain symmetric class of states. We can also generalize it for multi-qubit systems.

[4] Dong-Dong Dong , Li-Juan Li, Xue-Ke Song, Liu Ye, and Dong Wang, *Phys. Rev. A 110, 032420 (2024)*.

Volume of Assistance Contd..

- VoA can be generalized for three-qudit states by considering generalized concurrence of assistance (GCoA) for certain symmetric class of states. We can also generalize it for multi-qubit systems.
- A novel three-qubit GME measure minimum pairwise concurrence (MPC) has recently been introduced by Dong *et al.*[4] The measure is defined by

$$\mathcal{M}_{ABC} = \min\{\mathcal{C}_{A'B'}, \mathcal{C}_{A'C'}, \mathcal{C}_{B'C'}\}$$

where $\mathcal{C}_{A'B'} = \sqrt{\mathcal{C}^2_{AB} + \tau_{ABC}}$, $\mathcal{C}_{A'C'} = \sqrt{\mathcal{C}^2_{AC} + \tau_{ABC}}$ and $\mathcal{C}_{B'C'} = \sqrt{\mathcal{C}^2_{BC} + \tau_{ABC}}$ are called the pairwise concurrences of the respective bipartitions. τ_{ABC} is the three-tangle of the corresponding three-qubit pure state defined by $\tau_{ABC} = \mathcal{C}^2_{A|BC} - \mathcal{C}^2_{AB} - \mathcal{C}^2_{AC}$.

[4] Dong-Dong Dong , Li-Juan Li, Xue-Ke Song, Liu Ye, and Dong Wang, *Phys. Rev. A 110, 032420 (2024)*.

Contd..

- Consider the following states :

$$|\psi_3\rangle = \frac{1}{2}(|000\rangle + |100\rangle + \sqrt{2}\,|111\rangle)$$
$$|\psi_4\rangle = \frac{1}{2\sqrt{2}}(\sqrt{2}\,|000\rangle + |100\rangle + |101\rangle + 2\,|111\rangle)$$

Contd..

- Consider the following states :

$$|\psi_3\rangle = \frac{1}{2}(|000\rangle + |100\rangle + \sqrt{2}\,|111\rangle)$$
$$|\psi_4\rangle = \frac{1}{2\sqrt{2}}(\sqrt{2}\,|000\rangle + |100\rangle + |101\rangle + 2\,|111\rangle)$$

- We find that MPC fails to discriminate between these two states. In fact, $\mathcal{M}(|\psi_3\rangle) = \frac{1}{\sqrt{2}} = \mathcal{M}(|\psi_4\rangle)$. But according to VoA, $\overline{\mathcal{C}}_3(|\psi_3\rangle) = \frac{1}{\sqrt[3]{2}} \approx 0.794$ and $\overline{\mathcal{C}}_3(|\psi_4\rangle) = \frac{\sqrt[6]{15}}{2} \approx 0.785$.

Contd..

- Consider the following states :

$$|\psi_3\rangle = \frac{1}{2}(|000\rangle + |100\rangle + \sqrt{2}\,|111\rangle)$$
$$|\psi_4\rangle = \frac{1}{2\sqrt{2}}(\sqrt{2}\,|000\rangle + |100\rangle + |101\rangle + 2\,|111\rangle)$$

- We find that MPC fails to discriminate between these two states. In fact, $\mathcal{M}(|\psi_3\rangle) = \frac{1}{\sqrt{2}} = \mathcal{M}(|\psi_4\rangle)$. But according to VoA, $\overline{\mathcal{C}}_3(|\psi_3\rangle) = \frac{1}{\sqrt[3]{2}} \approx 0.794$ and $\overline{\mathcal{C}}_3(|\psi_4\rangle) = \frac{\sqrt[6]{15}}{2} \approx 0.785$.
- Actually, this is not an isolated example. There is a whole class of states where the MPC fails to discriminate. If we consider the following two classes of pure three-qubit states:

$$|\phi_1\rangle = \lambda_0\,|000\rangle + \lambda_1\,|100\rangle + \lambda_4\,|111\rangle$$
$$|\phi_2\rangle = \lambda_0\,|000\rangle + \mu\,|100\rangle + \lambda_2\,|101\rangle + \lambda_4\,|111\rangle$$

Contd..

where $\lambda_0, \lambda_1, \lambda_4 \geqslant 0$ satisfying $\lambda_0^2 + \lambda_1^2 + \lambda_4^2 = 1$; $\mu, \lambda_2 \geqslant 0$ satisfying $\lambda_0^2 + \mu^2 + \lambda_2^2 + \lambda_4^2 = 1$ and $\mu^2 + \lambda_2^2 = \lambda_1^2$.

Contd..

where $\lambda_0, \lambda_1, \lambda_4 \geqslant 0$ satisfying $\lambda_0^2 + \lambda_1^2 + \lambda_4^2 = 1$; $\mu, \lambda_2 \geqslant 0$ satisfying $\lambda_0^2 + \mu^2 + \lambda_2^2 + \lambda_4^2 = 1$ and $\mu^2 + \lambda_2^2 = \lambda_1^2$.

- A simple calculation will show MPC of both $|\phi_1\rangle$ and $|\phi_2\rangle$ are equal: $\mathcal{M}(|\phi_1\rangle) = \mathcal{M}(|\phi_2\rangle) = 2\lambda_0\lambda_4$. Nevertheless, according to VoA,

$$\overline{\mathcal{C}}_3(|\phi_1\rangle) = 2\lambda_4 \sqrt[3]{\lambda_0^2\sqrt{\lambda_0^2 + \lambda_1^2}}$$

$$\overline{\mathcal{C}}_3(|\phi_2\rangle) = 2\sqrt[3]{\lambda_0^2\lambda_4^2\sqrt{\lambda_0^2 + \mu^2}\sqrt{\lambda_2^2 + \lambda_4^2}}$$

clearly most of the time they are unequal.

Contd..

where $\lambda_0, \lambda_1, \lambda_4 \geqslant 0$ satisfying $\lambda_0^2 + \lambda_1^2 + \lambda_4^2 = 1$; $\mu, \lambda_2 \geqslant 0$ satisfying $\lambda_0^2 + \mu^2 + \lambda_2^2 + \lambda_4^2 = 1$ and $\mu^2 + \lambda_2^2 = \lambda_1^2$.

- A simple calculation will show MPC of both $|\phi_1\rangle$ and $|\phi_2\rangle$ are equal: $\mathcal{M}(|\phi_1\rangle) = \mathcal{M}(|\phi_2\rangle) = 2\lambda_0\lambda_4$. Nevertheless, according to VoA,

$$\overline{\mathcal{C}}_3(|\phi_1\rangle) = 2\lambda_4 \sqrt[3]{\lambda_0^2\sqrt{\lambda_0^2 + \lambda_1^2}}$$

$$\overline{\mathcal{C}}_3(|\phi_2\rangle) = 2\sqrt[3]{\lambda_0^2\lambda_4^2\sqrt{\lambda_0^2 + \mu^2}\sqrt{\lambda_2^2 + \lambda_4^2}}$$

clearly most of the time they are unequal.

- One of the essential use of multipartite entanglement is in creating singlets at a later time. In our work, we have taken this phenomena into account and introduce VoA as a bonafide measure of multipartite entanglement.

Role of Measurement- Imaginarity!

- Alice and Bob are given one of the following two states:

$$\rho_1^{AB} = \frac{1}{2}(|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|)$$
$$\rho_2^{AB} = \frac{1}{2}(|\phi^+\rangle\langle\phi^+| + |\psi^-\rangle\langle\psi^-|)$$

[5] K.-D. Wu, T. V. Kondra, S. Rana, C. M. Scandolo, G.-Y. Xiang, C.-F. Li, G.-C. Guo, and A. Streltsov, *Phys. Rev. Lett. 126, 090401 (2021)*

Role of Measurement- Imaginarity!

- Alice and Bob are given one of the following two states:

$$\rho_1^{AB} = \frac{1}{2}(|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|)$$
$$\rho_2^{AB} = \frac{1}{2}(|\phi^+\rangle\langle\phi^+| + |\psi^-\rangle\langle\psi^-|)$$

- The local POVM

$$M_1 = |\hat{+}\rangle\langle\hat{+}| \otimes |\hat{+}\rangle\langle\hat{+}| + |\hat{-}\rangle\langle\hat{-}| \otimes |\hat{-}\rangle\langle\hat{-}|$$
$$M_2 = |\hat{+}\rangle\langle\hat{+}| \otimes |\hat{-}\rangle\langle\hat{-}| + |\hat{-}\rangle\langle\hat{-}| \otimes |\hat{+}\rangle\langle\hat{+}|$$

can perfectly distinguish them, where $|\hat{\pm}\rangle = \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}$.

[5] K.-D. Wu, T. V. Kondra, S. Rana, C. M. Scandolo, G.-Y. Xiang, C.-F. Li, G.-C. Guo, and A. Streltsov, *Phys. Rev. Lett. 126, 090401 (2021)*

Role of Measurement- Imaginarity!

- Alice and Bob are given one of the following two states:

$$\rho_1^{AB} = \frac{1}{2}(|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|)$$
$$\rho_2^{AB} = \frac{1}{2}(|\phi^+\rangle\langle\phi^+| + |\psi^-\rangle\langle\psi^-|)$$

- The local POVM

$$M_1 = |\hat{+}\rangle\langle\hat{+}| \otimes |\hat{+}\rangle\langle\hat{+}| + |\hat{-}\rangle\langle\hat{-}| \otimes |\hat{-}\rangle\langle\hat{-}|$$
$$M_2 = |\hat{+}\rangle\langle\hat{+}| \otimes |\hat{-}\rangle\langle\hat{-}| + |\hat{-}\rangle\langle\hat{-}| \otimes |\hat{+}\rangle\langle\hat{+}|$$

can perfectly distinguish them, where $|\hat{\pm}\rangle = \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}$.

- It turns out no real(?) operation can distinguish them with any finite probability[5].

[5] K.-D. Wu, T. V. Kondra, S. Rana, C. M. Scandolo, G.-Y. Xiang, C.-F. Li, G.-C. Guo, and A. Streltsov, *Phys. Rev. Lett. 126, 090401 (2021)*

Contd..

- Any state that can be represented as :

$$|\psi_r\rangle = \sum_i r_i |i\rangle$$

where $r_i \in \mathbb{R}\ \forall i$ ($\mathbb{R}$ being the set of all real numbers) satisfying $\sum_i r_i^2 = 1$, with respect to the computational basis $\{|i\rangle\langle i|\}$, is called a real state.

Contd..

- Any state that can be represented as :

$$|\psi_r\rangle = \sum_i r_i |i\rangle$$

where $r_i \in \mathbb{R}\ \forall i$ ($\mathbb{R}$ being the set of all real numbers) satisfying $\sum_i r_i^2 = 1$, with respect to the computational basis $\{|i\rangle\langle i|\}$, is called a real state.

- A state is non-real or imaginary if it can be written as:

$$|\psi_c\rangle = \sum_i c_i |i\rangle$$

where $c_i \in \mathbb{C}$ ($\mathbb{C}$ being the set of complex numbers) satisfying $\sum_i |c_i|^2 = 1$, with at least one j such that $\mathrm{Im}\{c_j\} \neq 0$.

Contd..

- Like resource theory of entanglement or coherence, maximal imaginary state exists and is given by :

$$\left|\hat{+}\right\rangle = \frac{|0\rangle + \mathrm{i}\,|1\rangle}{\sqrt{2}}$$

where $\mathrm{i} = \sqrt{-1}$. Note that the state $\left|\hat{+}\right\rangle$ can also be converted into $\left|\hat{-}\right\rangle = (|0\rangle - \mathrm{i}\,|1\rangle)/\sqrt{2}$ by a real orthogonal matrix. The state $\left|\hat{-}\right\rangle$ is in fact also maximally resourceful.

- To quantify imaginarity, there are several measures of imaginarity like ‘Robustness of Imaginarity’, ‘Geometric Imaginarity’ etc. However, we will only consider ‘Robustness of Imaginarity’ (RoI) in our work.

Contd..

- For any state ρ, its RoI is defined as:

$$\begin{aligned}\mathcal{R}(\rho) =& \min_{\tau}\{s \geqslant 0 : \frac{\rho + s\tau}{1+s} \text{ is a real state}\} \\ =& \frac{1}{2}\|\rho - \rho^{\mathsf{T}}\|_1\end{aligned}$$

where $\|M\|_1 = \operatorname{Tr}\sqrt{M^\dagger M}$.

Contd..

- For any state ρ, its RoI is defined as:

$$\mathcal{R}(\rho) = \min_{\tau}\{s \geqslant 0 : \frac{\rho + s\tau}{1+s} \text{ is a real state}\}$$
$$= \frac{1}{2}\|\rho - \rho^{\mathsf{T}}\|_1$$

where $\|M\|_1 = \mathrm{Tr}\sqrt{M^\dagger M}$.

- To concentrate entanglement of the genuinely entangled $\lambda_0 |000\rangle + \lambda_4 |111\rangle$ into any bi-partition, one needs to measure the third party in the eigenbasis of Pauli σ_x or σ_y *i.e.*, by maximal coherent state. Therefore the basis does not depend on the specific parametric values of the state.

Contd..

- But if we are given a slice state $|\psi_s\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle$ such that $\mathbf{C}_{AB} = 0, \mathbf{C}_{AC} = 0, \mathbf{C}_{BC} \neq 0$ and want to concentrate entanglement between B and C, then one has to do local measurement setup depending on the parametric values.

Contd..

- But if we are given a slice state $|\psi_s\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle$ such that $\mathbf{C}_{AB} = 0, \mathbf{C}_{AC} = 0, \mathbf{C}_{BC} \neq 0$ and want to concentrate entanglement between B and C, then one has to do local measurement setup depending on the parametric values.
- Consider the real projectors:

$$|+k_\alpha\rangle = \cos\alpha |0\rangle + \sin\alpha |1\rangle$$
$$|-k_\alpha\rangle = \sin\alpha |0\rangle - \cos\alpha |1\rangle$$

Contd..

- But if we are given a slice state $|\psi_s\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle$ such that $\mathbf{C}_{AB} = 0, \mathbf{C}_{AC} = 0, \mathbf{C}_{BC} \neq 0$ and want to concentrate entanglement between B and C, then one has to do local measurement setup depending on the parametric values.
- Consider the real projectors:

$$|+k_\alpha\rangle = \cos\alpha |0\rangle + \sin\alpha |1\rangle$$
$$|-k_\alpha\rangle = \sin\alpha |0\rangle - \cos\alpha |1\rangle$$

- The optimal is achieved for $\alpha = \tan^{-1}\left(\frac{\sqrt{1-\lambda_4^2}+\lambda_1}{\lambda_0}\right)$–a clear dependence on the specific state.
- But if we use the maximally imaginary basis, we can let go of the dependence on the parameters and therefore, establishes a clear advantage over the real basis.

Contd..

- Again consider a subclass of the slice state:

$$|\psi_s(a)\rangle = b\,|000\rangle + b\,|100\rangle + a\,|111\rangle \tag{1}$$

satisfying $a, b \geqslant 0$ and $a^2 + 2b^2 = 1$.

Contd..

- Again consider a subclass of the slice state:

$$|\psi_s(a)\rangle = b\,|000\rangle + b\,|100\rangle + a\,|111\rangle \tag{1}$$

satisfying $a, b \geqslant 0$ and $a^2 + 2b^2 = 1$.

- Consider a class of generic qubit projective measurement basis:

$$|+k\rangle = \cos\alpha\,|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\alpha\,|1\rangle$$
$$|-k\rangle = \mathrm{e}^{-\mathrm{i}\beta}\sin\alpha\,|0\rangle - \cos\alpha\,|1\rangle$$

such that $\alpha \in [0, \pi/4]$, $\beta \in [0, \pi)$, which the reduces to the real projective basis mentioned before for $\beta = 0$.

Contd..

- Again consider a subclass of the slice state:

$$|\psi_s(a)\rangle = b\,|000\rangle + b\,|100\rangle + a\,|111\rangle \tag{1}$$

satisfying $a, b \geqslant 0$ and $a^2 + 2b^2 = 1$.

- Consider a class of generic qubit projective measurement basis:

$$|+k\rangle = \cos\alpha\,|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\alpha\,|1\rangle$$
$$|-k\rangle = \mathrm{e}^{-\mathrm{i}\beta}\sin\alpha\,|0\rangle - \cos\alpha\,|1\rangle$$

such that $\alpha \in [0, \pi/4]$, $\beta \in [0, \pi)$, which the reduces to the real projective basis mentioned before for $\beta = 0$.

- We find that the entanglement concentration yield is always higher whenever we take $\beta \neq 0$ compared the corresponding real basis(*i.e.*, for $\beta = 0$).

Contd..

- Next we consider a three-qubit swapping scenario where the goal is to create genuinely entangled state first and then concentrate entanglement between two out of the three parties. The motivation for doing this will be clear once we show that it helps in quantum entanglement percolation.

Contd..

- Next we consider a three-qubit swapping scenario where the goal is to create genuinely entangled state first and then concentrate entanglement between two out of the three parties. The motivation for doing this will be clear once we show that it helps in quantum entanglement percolation.
- Conventionally the three-qubit swapping has always been performed by the three-qubit genuine maximally entangled basis:

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left|000 \pm 111\right\rangle, \frac{1}{\sqrt{2}}\left|001 \pm 110\right\rangle \\ &\frac{1}{\sqrt{2}}\left|010 \pm 101\right\rangle, \frac{1}{\sqrt{2}}\left|100 \pm 011\right\rangle \end{aligned} \tag{2}$$

- The genuinely entangled state will be created among three parties who initially shares a bipartite pure entangled state of the form: $|\phi\rangle = \sqrt{\phi_0}\,|00\rangle + \sqrt{\phi_1}\,|11\rangle$ with $\phi_0 > \phi_1 > 0$ and $\phi_0 + \phi_1 = 1$.

Contd..

- Using this basis results in a yield $E^{i|j}_{2,GHZ} = 2\phi_1^2(\phi_1 + 3\phi_0)$ where $i, j \in \{A, B, C\}$.

Figure: Three-qubit entanglement swapping

- But here we consider another imaginary basis:

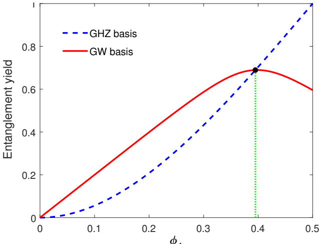
$$
\begin{aligned}
|G_1\rangle &= \frac{1}{\sqrt{5}}(|000\rangle + |110\rangle + |101\rangle + |011\rangle + |111\rangle) \\
|G_2\rangle &= \frac{1}{\sqrt{5}}(|000\rangle + \alpha\,|110\rangle + \alpha^2\,|101\rangle + \alpha^3\,|011\rangle + \alpha^4\,|111\rangle) \\
|G_3\rangle &= \frac{1}{\sqrt{5}}(|000\rangle + \alpha^2\,|110\rangle + \alpha^4\,|101\rangle + \alpha\,|011\rangle + \alpha^3\,|111\rangle) \\
|G_4\rangle &= \frac{1}{\sqrt{5}}(|000\rangle + \alpha^3\,|110\rangle + \alpha\,|101\rangle + \alpha^4\,|011\rangle + \alpha^2\,|111\rangle) \\
|G_5\rangle &= \frac{1}{\sqrt{5}}(|000\rangle + \alpha^4\,|110\rangle + \alpha^3\,|101\rangle + \alpha^2\,|011\rangle + \alpha\,|111\rangle) \\
|W_1\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \\
|W_2\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + \omega\,|010\rangle + \omega^2\,|001\rangle) \\
|W_3\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + \omega^2\,|010\rangle + \omega\,|001\rangle)
\end{aligned}
$$

Contd..

where ω is the cube root of unity and α is the fifth root of unity. We call it the GW basis.

- This results in the entanglement yield:

$$E^{A|B}_{2,GW} = 1 - \phi_0^2\phi_1 - \sqrt{(\phi_0^3 + \phi_1^3 + 3\phi_0\phi_1^2)^2 - 4\phi_0^2\phi_1^2(2 - 3\phi_0\phi_1)}$$



Contd..

- One can see that the GW basis always yields higher than the GHZ basis whenever $0 \leqslant \phi_1 \leqslant 0.39493$.
- The GW-basis looks fairly asymmetric. However, the singlet conversion probability for all the basis states are identical irrespective of the parties, *i.e.*, $E_2^{A|B}(|\psi\rangle) = E_2^{A|C}(|\psi\rangle) = E_2^{B|C}(|\psi\rangle)$, $\forall \, |\psi\rangle \in \{|G_i\rangle \, , |W_j\rangle : i = 1, 2, 3, 4, 5; j = 1, 2, 3\}$

Contd..

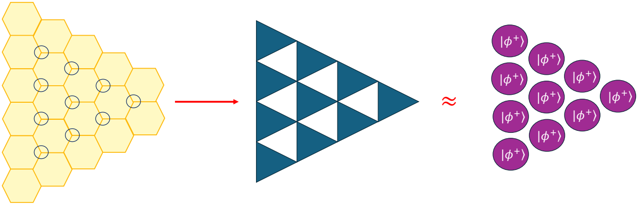


Figure: Three-qubit entanglement swapping

- Consider an entanglement percolation scenario where the goal is to lower the requirement of entanglement for percolation by converting lattices.

Contd..

- Specifically we consider[6],[7] where a honeycomb or hexagonal lattice is converted into triangular lattices so that the percolation threshold probability is lower.
- It has been shown that swapping in GHZ basis does not result in any improvement in the requirement of entanglement. However, the requirement can be reduced by 0.1 ebit or 10.6% by measuring in GW-basis.
- Finally, we do not know whether the GW basis is optimal. However, We searched numerically and could not find any other basis that performs better than GW basis.

---

[6] A. Acin, J. I. Cirac, and M. Lewenstein, Nat. Phys. 3, 256 (2007)

[7] S. Khanna, S. Halder, and U. Sen, Phys. Rev. A 109, 012419 (2024)

Summary

- We observe that structure of multipartite system is quite different from that of bipartite system.
- A new measure (VoA) is formed that proves in certain cases to be more robust than others.
- The role of measurement with a real or imaginary basis is an important issue to deal with multipartite systems.

Thank you