Discrete-Time Open Quantum Walks for Vertex Ranking in Graphs

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by

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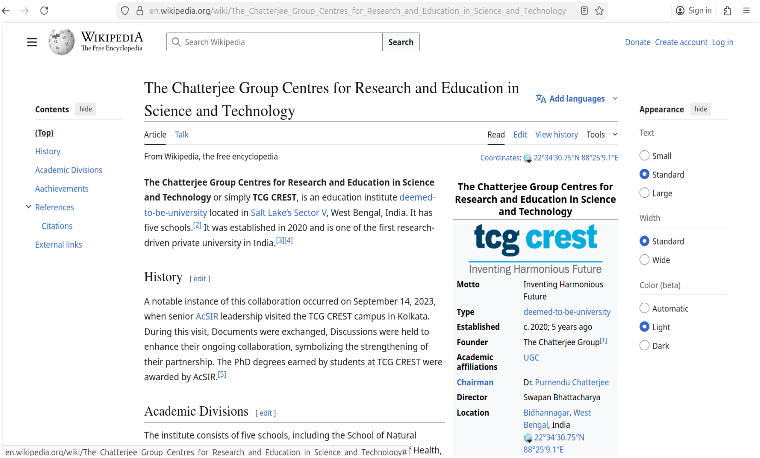
Webpages and Hyperlinks



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Contents hide

(Top)
History
Academic Divisions
Aachievements
References
Citations
External links

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Article Talk | Read Edit View history Tools

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Coordinates: 22°34′30.75″N 88°25′9.1″E

The Chatterjee Group Centres for Research and Education in Science and Technology or simply **TCG CREST**, is an education institute deemed-to-be-university located in Salt Lake's Sector V, West Bengal, India. It has five schools.[2] It was established in 2020 and is one of the first research-driven private university in India.[3][4]

History [edit]

A notable instance of this collaboration occurred on September 14, 2023, when senior AcSIR leadership visited the TCG CREST campus in Kolkata. During this visit, Documents were exchanged, Discussions were held to enhance their ongoing collaboration, symbolizing the strengthening of their partnership. The PhD degrees earned by students at TCG CREST were awarded by AcSIR.[5]

Academic Divisions [edit]

The institute consists of five schools, including the School of Natural ... Health,

The Chatterjee Group Centres for Research and Education in Science and Technology

tcg crest
Inventing Harmonious Future

Motto	Inventing Harmonious Future
Type	deemed-to-be-university
Established	c. 2020; 5 years ago
Founder	The Chatterjee Group[1]
Academic affiliations	UGC
Chairman	Dr. Purnendu Chatterjee
Director	Swapan Bhattacharya
Location	Bidhannagar, West Bengal, India 22°34′30.75″N 88°25′9.1″E

Appearance hide

Text: Small / Standard / Large
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Color (beta): Automatic / Light / Dark

en.wikipedia.org/wiki/The_Chatterjee_Group_Centres_for_Research_and_Education_in_Science_and_Technology#

Figure: A snapshot of the Wikipedia page of TCG CREST. The blue colored texts contains the hyperlinks to take to reader to another Wikipedia page.

The webgraphs

Webgraph: Set of directed links between pages of WWW

- A webpage consists of data and hyperlinks.[1]
- A hyperlink takes a reader from one webpage to another.

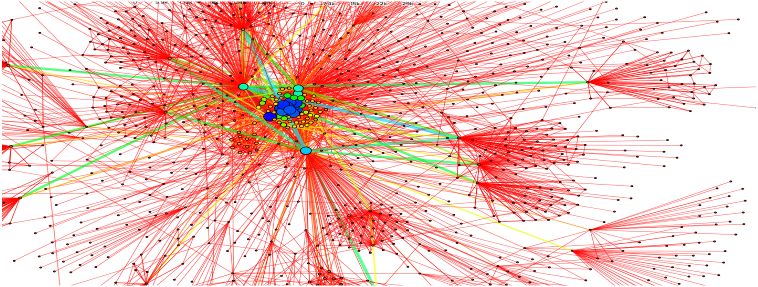

Figure: A snapshot of the web-BerkStan network which represents the hyperlink structure of web pages from Berkeley and Stanford domains.

[1] Ryan A. Rossi and Nesreen K. Ahmed. “The Network Data Repository with Interactive Graph Analytics and Visualization”. In: *AAAI*. 2015. URL: https://networkrepository.com.

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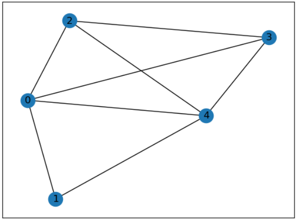
Graphs

- A graph $G = (V(G), E(G))$ consists of a set of vertices $V(G)$ and a set of edges $E(G) \subset V(G) \times V(G)$.
- Pictorially, a vertex is represented by a dot (•) and an edge is represented by a line or an arch joining two dots.
- The edge (u, v) is incident to u and v. Also, u and v are the neighbors of (adjacent to) each other.
- The degree of v is d_v which is the number of vertices u adjacent to v.
- The set of the adjacent vertices of v is the neighbourhood of v, denoted by $\text{nbd}(v)$.

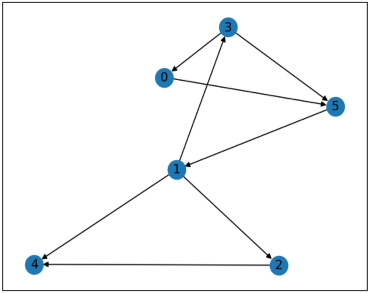


Vertices	degree	neighbors
0	4	[1, 2, 3, 4]
1	2	[0, 4]
2	3	[0, 3, 4]
3	3	[0, 2, 4]
4	4	[0, 1, 2, 3]

Directed graph

- $\overrightarrow{(u,v)}$ is a directed edge or link, coming into v from u.
- In a directed graph $\overrightarrow{G}$ all the edges are directed.
- $\overrightarrow{(u,v)}$ is an inlink of v. The number of inlinks of v is called the indegree of v which is denoted by $d_v^{(i)}$.
- The set of vertices u for which there is an inlink $\overrightarrow{(u,v)}$ to v is denoted by $B(v)$.
- $\overrightarrow{(v,u)}$ is an outlinks of v. The number of outlinks of v is its outdegree, which is denoted by $d_v^{(o)}$.
- If $d_v^{(o)} = 0$, then v is called a pendent vertex or dangling node.
- The set of vertices u for which there is an outlink $\overrightarrow{(v,u)}$ from v is denoted by $\text{Nbd}(v)$.

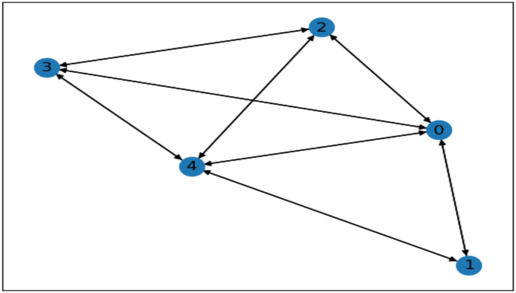
Example directed graph



Vertices	In-degree	In-nbd	Out-degree	Out-nbd
0	1	[3]	1	[5]
1	1	[5]	3	[2, 3, 4]
2	1	[1]	1	[4]
3	1	[1]	2	[5, 0]
4	2	[1, 2]	0	[]
5	2	[0, 3]	1	[1]

From an undirected graph to a directed graph

- An undirected edge (u, v) is a combination of two directed edges, $\overrightarrow{(u, v)}$ and $\overrightarrow{(v, u)}$.
- An undirected graph G can be converted to a directed graph $\overrightarrow{G}$ by adding two different orientations on its edges.
- Let G be an undirected graph and $\overline{G}$ be the corresponding directed graph, then $\text{nbd}(v) = B(v) = \text{Nbd}(v)$ and $d_v = d_v^{(o)} = d_v^{(i)}$, for every vertex v.



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PageRank

Definition

The PageRank[2] of a vertex v in $\overrightarrow{G}$ is

$$r(v) = \sum_{u \in B(v)} \frac{r(u)}{d_u^o}, \tag{1}$$

In an iterative way,

$$r_{k+1}(v) = \sum_{u \in B(v)} \frac{r_k(u)}{d_u^o}. \tag{2}$$

Initially, $r_0(v) = \frac{1}{n}$ for all vertices v, where n is the total number of vertices in $\overrightarrow{G}$.

$$r(v) = \lim_{k \to \infty} r_k(v). \tag{3}$$

[2] Lawrence Page et al. *The PageRank citation ranking: Bringing order to the web.* Tech. rep. Stanford infolab, 1999.

Graphs and matrices

Adjacency matrix of a graph

$A = (a_{u,v})_{n \times n}$, where

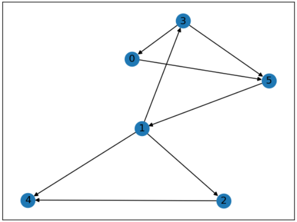
$$a_{u,v} = \begin{cases} 1 & \text{if } \overrightarrow{(u,v)} \in E(\overrightarrow{G}); \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Hyperlink matrix of a graph

$H = (h_{u,v})_{n \times n}$, where

$$h_{u,v} = \begin{cases} \frac{1}{d_u^o} & \text{if } \overrightarrow{(u,v)} \in E(\overrightarrow{G}); \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Example



The adjacency and hyperlink matrix of the above graph:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Stochastically adjustment

If v is a dangling node, then we replace the matrix H by

$$S = H + \frac{1}{n} a e^{\dagger},$$

where e is the all 1 vector and the vector a consists of the elements $a_v = 1$ if v is a dangling node and 0 otherwise.

Vertex 4 is dangling

$$a = (0, 0, 0, 0, 1, 0)^t$$
$$e = (1, 1, 1, 1, 1, 1)^t$$
$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6)$$

Primitivity adjustment and restart in surfing

Without following the hyperlink method of surfing a surfer may teleport to a new destination with probability α, where (s)he begins hyperlink surfing again until the next teleportation. Hence, we update the matrix S as the Google matrix

$$\mathcal{G} = \alpha S + \frac{(1-\alpha)}{n} e e^{\dagger}. \tag{7}$$

Usually, $\alpha = 0.85$.

Google matrix

$$\mathcal{G} = 0.85 \times \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{0.15}{6} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Google matrix and its properties

Google Matrix

$$\mathcal{G} = \begin{bmatrix} \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{7}{8} \\ \frac{1}{40} & \frac{1}{40} & \frac{37}{120} & \frac{37}{120} & \frac{37}{120} & \frac{1}{40} \\ \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{7}{8} & \frac{1}{40} \\ \frac{9}{20} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{9}{20} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{40} & \frac{7}{8} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} \end{bmatrix} \tag{8}$$

Properties of the Google matrix of any graph

1. $\mathcal{G}$ is a stochastic matrix. Row sum of each row is 1.
2. $\mathcal{G}$ is irreducible. The corresponding weighted directed graph of the matrix is strongly connected.
3. $\mathcal{G}$ is aperiodic.
4. $\mathcal{G}$ is primitive. All the entries are positive.
5. Rank of $\mathcal{G}$ is always more than 1.

PageRank is a limiting probability distribution of a random walk

Initial PageRank vector

$$\Pi^{(0)} = \frac{1}{n}(1, 1, \dots 1(n\text{-times})).$$

Time evolution of random walk

The probability vector at $(k+1)$-th iteration is $\Pi^{(k+1)} = \Pi^{(k)}\mathcal{G}$, for $k = 0, 1, 2, \dots$.

Page rank

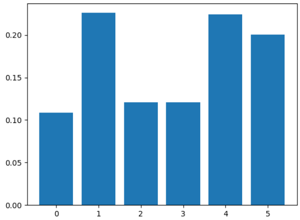
The above properties of $\mathcal{G}$ ensures that the sequence of probability vectors $\Pi^{(k+1)} = \Pi^{(k)}\mathcal{G}$, for $k = 0, 1, 2, \dots$ converges. The PageRank vector[3] is given by

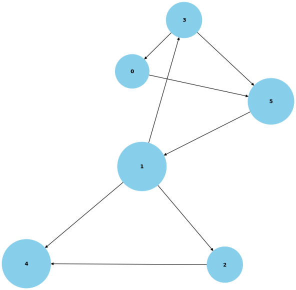
$$\Pi = \lim_{k \to \infty} \Pi^{(k)}. \qquad (9)$$

[3] Amy N Langville and Carl D Meyer. *Google's PageRank and beyond: The science of search engine rankings.* Princeton University Press, 2006.

PageRank of the example graph

PageRank Vertices	
0	0.108500
1	0.225900
2	0.120600
3	0.120600
4	0.223900
5	0.200400





Desired properties of quantum PageRank

A qPageRank should satisfy the following assumptions[4]:

1. The classical PageRank is defined on directed graphs. Hence, the qPageRank should also be defined on directed graphs.
2. The values of qPageRank of the vertices should be in $[0, 1]$. Also, the sum of all qPageRanks in a graph is 1.
3. The qPageRank must admit a quantized Markov Chain description.
4. The classical algorithm to compute qPageRank of the vertices in a graph belongs to the computational complexity class BQP. This criterion can be further restricted to enforce that the computational complexity of qPageRank must be P.

[4] Giuseppe Davide Paparo and Miguel Angel Martin-Delgado. "Google in a quantum network". In: *Scientific Reports* 2.1 (2012), p. 444.

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Difficulties in replacing the random walks by quantum walks in PageRank algorithm

- The dynamics of quantum walks and random walks are different. Therefore, their limiting probability distributions converges in differently.
- Different models of the quantum walks has different limitations. For instance,
  - Continuous-time quantum walks are defined for any undirected graphs.
  - Discrete-time quantum walks are mostly studied for regular graphs, such as infinite path graphs, cycle graphs, etc.
  - The Szegedy quantum walk is originally defined for bipartite graphs.

  We need a model of quantum walks which is fit for **any directed graph**, which is not readily available.
- The computation of classical PageRank includes stochastically adjustment and primitivity adjustment due to their physical needs. These physical requirements should also be fulfilled by the qPageRank.

Quantum channels in higher dimensional

Quantum channel and Kraus operators[5,6]

- **Completeness condition** $\sum_{i=1}^{d} K_i^\dagger K_i = I_n$.
- **Evolution of state** $\Psi(\rho) = \sum_{i=1}^{d} K_i \rho K_i^\dagger$.

Unitary operators in any dimension: Weyl operators[7,8]

$$U_{r,s} = \sum_{i=0}^{n-1} \exp\left(\frac{2\pi\iota}{n}\right)^{ir} |i\rangle \langle i \oplus s| \text{ for } 0 \leq r, s \leq n-1. \quad (10)$$

[5] ECG Sudarshan, PM Mathews, and Jayaseetha Rau. "Stochastic dynamics of quantum-mechanical systems". In: *Physical Review* 121.3 (1961), p. 920.

[6] Karl Kraus et al. *States, Effects, and Operations Fundamental Notions of Quantum Theory: Lectures in Mathematical Physics at the University of Texas at Austin*. Springer, 1983.

[7] Charles H Bennett et al. "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels". In: *Physical Review Letters* 70.13 (1993), p. 1895.

[8] Supriyo Dutta, Subhashish Banerjee, and Monika Rani. "Qudit states in noisy quantum channels". In: *Physica Scripta* 98.11 (2023), p. 115113.

Weyl operators for the example graph

$$U_{0,5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, U_{1,2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^3 \\ \omega^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^5 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$U_{1,3} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^2 \\ \omega^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^5 & 0 & 0 & 0 \end{bmatrix}, U_{1,4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \\ \omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^5 & 0 & 0 \end{bmatrix}.$$

Corresponding to all eight edges we can construct eight Weyl operators, which are

$$U_{0,5}, U_{1,2}, U_{1,3}, U_{1,4}, U_{2,4}, U_{3,5}, U_{3,0}, \text{ and } U_{5,1}.$$

Coin operators of quantum walk

- Let v be a vertex with outdegree $d_v^{(o)}$ in a directed graph $\overrightarrow{G}$.
- At the next time-step the walker will be either at v or any of the $d_v^{(o)}$ vertices via $\overrightarrow{(v, u_1)}, \overrightarrow{(v, u_2)}, \dots \overrightarrow{(v, u_{d_v^{(o)}})}$.
- Construct $(d_v^{(o)} + 1)$ Weyl operators $I_n, U_{v,u_1}, \dots, U_{v,u_{d_v^{(o)}}}$.
- As U_{v,u_i} are unitary operators for $i = 1, 2, \dots d_v^{(o)}$, we have

$$I_n I_n^\dagger + U_{v,u_1}^\dagger U_{v,u_1} + \dots + U_{v,u_{d_v^{(o)}}}^\dagger U_{v,u_{d_v^{(o)}}} = (d_v^{(o)} + 1) I_n. \quad (11)$$

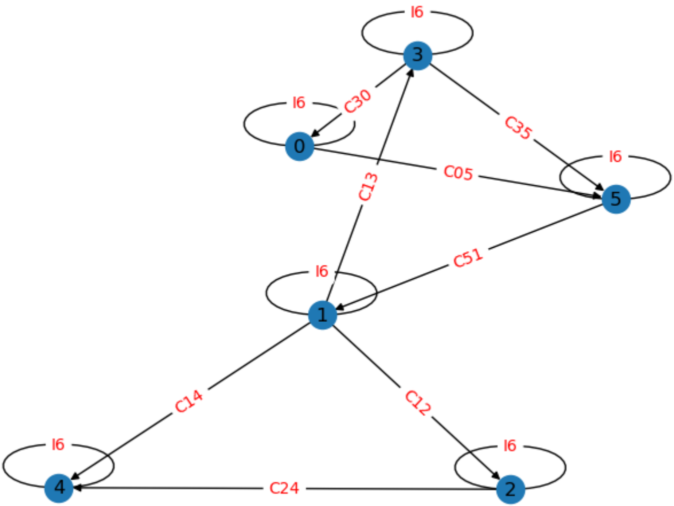
- We can construct $(d_v^{(o)} + 1)$ coin operators for v which are

$$C_{v,v} = \frac{1}{\sqrt{(d_v^{(o)} + 1)}} I_n, \text{ and } C_{v,u_i} = \frac{1}{\sqrt{(d_v^{(o)} + 1)}} U_{v,u_i}, \quad (12)$$

for $i = 1, 2, \dots d_v^{(o)}$.
- The set of coin operators corresponding to the vertex v is

$$\mathcal{C}_v = \left\{ C_{v,u} = \frac{1}{\sqrt{(d_v^{(o)} + 1)}} U_{v,u}, u \in \text{Nbd}(v) \right\} \cup \{C_{v,v}\}. \quad (13)$$

Coin operators on the example graph



Shift operators of quantum walk

- Corresponding to vertex v we construct a basis state vector

$$|v\rangle = (0, 0, \dots 0, 1(v\text{-th position}), 0, \dots 0).$$

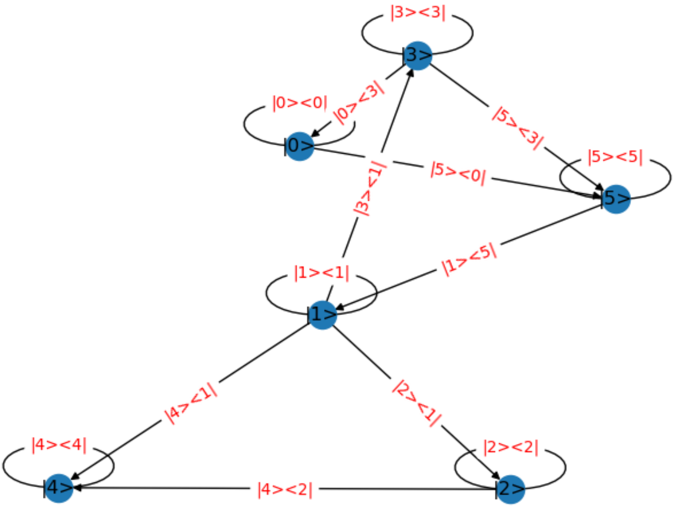
- Corresponding to an edge $\overrightarrow{(v, u_i)}$ we construct a shift operator $S_{v,u_i} = |u_i\rangle \langle v|$ for $i = 1, 2, \dots d_v^{(o)}$.
- If the walker do not move from vertex v at time t, we consider $S_{v,v} = |v\rangle \langle v|$ as a shift operator.
- The set of all shift operators at vertex v is

$$\mathcal{S}_v = \{S_{v,u}, u \in \text{Nbd}(v)\} \cup \{S_{v,v}\}. \tag{14}$$

Coin and Shift operators for our example graph

V	$d_v^{(o)}$	Out-nbd	Coin operators	Shift operators
0	1	[5]	$\left[\frac{I_6}{\sqrt{2}}, \frac{U_{0,5}}{\sqrt{2}}\right]$	$[S_{0,0}, S_{0,5}]$
1	3	[2, 3, 4]	$\left[\frac{I_6}{2}, \frac{U_{1,2}}{2}, \frac{U_{1,3}}{2}, \frac{U_{1,4}}{2}\right]$	$[S_{1,1}, S_{1,2}, S_{1,3}, S_{1,4}]$
2	1	[4]	$\left[\frac{I_6}{\sqrt{2}}, \frac{U_{2,4}}{\sqrt{2}}\right]$	$[S_{2,2}, S_{2,4}]$
3	2	[5, 0]	$\left[\frac{I_6}{\sqrt{3}}, \frac{U_{3,5}}{\sqrt{3}}, \frac{U_{3,0}}{\sqrt{3}}\right]$	$[S_{3,3}, S_{3,5}, S_{3,0}]$
4	0	[]	$[I_6]$	$[S_{4,4}]$
5	1	[1]	$\left[\frac{I_6}{\sqrt{2}}, \frac{U_{5,1}}{\sqrt{2}}\right]$	$[S_{5,5}, S_{5,1}]$

Shift operators for example graph



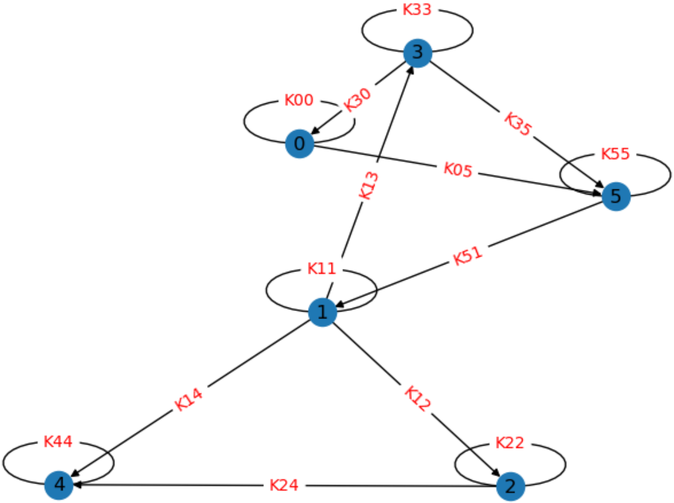
Completeness criterion for $K_{v,u} = C_{v,u} \otimes S_{v,u}$

$$
\begin{aligned}
&\sum_{v\in V(G)} \sum_{u\in \mathrm{Nbd}(v)} K_{v,u}^{\dagger} K_{v,u} \\
&= \sum_{v\in V(G)} \sum_{u\in \mathrm{Nbd}(v)\cup\{v\}} \left(C_{v,u}^{\dagger} \otimes S_{v,u}^{\dagger}\right)\left(C_{v,u} \otimes S_{v,u}\right) \\
&= \sum_{v\in V(G)} \sum_{u\in \mathrm{Nbd}(v)\cup\{v\}} \left(C_{v,u}^{\dagger} \otimes |v\rangle\langle u|\right)\left(C_{v,u} \otimes |u\rangle\langle v|\right) \\
&= \sum_{v\in V(G)} \left(\sum_{u\in \mathrm{Nbd}(v)\cup\{v\}} C_{v,u}^{\dagger} C_{v,u}\right) \otimes |v\rangle\langle u|u\rangle\langle v| \\
&= \sum_{v\in V(G)} I_n \otimes |v\rangle\langle v| \qquad \text{[using equation (11)]} \\
&= I_n \otimes I_n = I_{n^2}.
\end{aligned}
\tag{15}
$$

Thus, a set of Kraus operators on the graph $\overrightarrow{G}$ is

$$
\mathcal{K} = \{K_{v,u} : v \in V(G), u \in \mathrm{Nbd}(v) \cup \{v\}\}. \tag{16}
$$

Kraus operators for quantum walks



Evolution of the state of quantum walker I

- The initial density matrix as

$$\rho^{(0)} = \sum_{v \in V(G)} \frac{1}{n^2} I_n \otimes |v\rangle\langle v|. \tag{17}$$

- Probability of getting the walker at vertex v is

$$\begin{aligned} p_v^{(0)} &= \text{trace}\left((I \otimes |v\rangle\langle v|)\rho^{(0)}(I \otimes |v\rangle\langle v|)\right) \\ &= \text{trace}(\frac{1}{n^2} I_n \otimes |v\rangle\langle v|) = \frac{1}{n^2}\text{trace}(I_n) \times 1 = \frac{1}{n}. \end{aligned} \tag{18}$$

- State of the walker at time t is

$$\rho^{(t)} = \sum_{v \in V(G)} \rho_v^{(t)} \otimes |v\rangle\langle v|. \tag{19}$$

where $\rho_v^{(t)}$ is not necessarily as density matrix.

Evolution of the state of quantum walker II

- Probability of getting the walker at the vertex v at time t is

$$p_v^{(t)} = \text{trace}\left((I \otimes |v\rangle\langle v|)\rho^{(t)}(I \otimes |v\rangle\langle v|)\right)$$
$$= \text{trace}\left(\rho_v^{(t)} \otimes |v\rangle\langle v|\right) = \text{trace}\left(\rho_v^{(t)}\right) \times 1 = \text{trace}\left(\rho_v^{(t)}\right). \tag{20}$$

- The state of the walker at time $(t+1)$ is 1

$$\rho^{(t+1)} = \Psi_K(\rho^{(t)}) = \sum_{v \in V(G)} \sum_{u \in \text{Nbd}(v) \cup \{v\}} K_{v,u}\rho_v^{(t)} \otimes |v\rangle\langle v| K_{v,u}^\dagger$$
$$= \sum_{v \in V(G)} \sum_{u \in \text{Nbd}(v) \cup \{v\}} C_{v,u}\rho_v^{(t)} C_{v,u}^\dagger \otimes S_{v,u}|v\rangle\langle v| S_{v,u}^\dagger$$
$$= \sum_{v \in V(G)} \sum_{u \in \text{Nbd}(v) \cup \{v\}} C_{v,u}\rho_v^{(t)} C_{v,u}^\dagger \otimes |u\rangle\langle v|v\rangle\langle v|v\rangle\langle u|$$
$$= \sum_{v \in V(G)} \sum_{u \in \text{Nbd}(v) \cup \{v\}} C_{v,u}\rho_v^{(t)} C_{v,u}^\dagger \otimes |u\rangle\langle u|. \tag{21}$$

Kraus operators to teleport the walker

- As $S_{uv} = |u\rangle\langle v|$, we have

$$S_{uv}|v\rangle\langle v|S_{uv}^{\dagger} = |u\rangle\langle v|v\rangle\langle v|v\rangle\langle u| = |u\rangle\langle u|. \tag{22}$$

Also,

$$\sum_{v\in V(G)} S_{uv}^{\dagger}S_{uv} = \sum_{v\in V(G)} |v\rangle\langle v| = I_n. \tag{23}$$

- There are n^2 operators

$$\mathcal{D} = \left\{D_{u,v} : D_{u,v} = \frac{1}{\sqrt{n}}(I_n \otimes S_{u,v})\right\}. \tag{24}$$

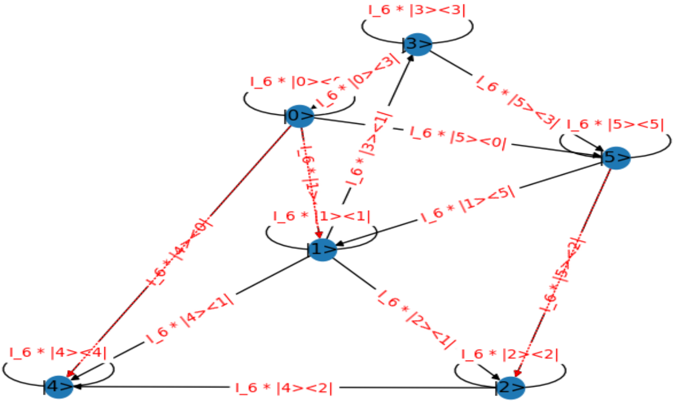
- The set $\mathcal{D}$ is a set of Kraus operators because

$$\sum_{v\in V(G)} D_{u,v}^{\dagger}D_{u,v} = \sum_{v\in V(G)} \frac{1}{\sqrt{n}}(I_n \otimes S_{u,v})^{\dagger}\frac{1}{\sqrt{n}}(I_n \otimes S_{u,v})$$

$$= \frac{1}{n}\left(I_n \otimes \sum_{v\in V(G)} S_{u,v}^{\dagger}S_{u,v}\right) = \frac{1}{n}I_n \otimes I_n = \frac{I_{n^2}}{n}$$

or $$\sum_{u\in V(G)}\sum_{v\in V(G)} D_{u,v}^{\dagger}D_{u,v} = \sum_{u\in V(G)} \frac{I_{n^2}}{n} = I_{n^2}. \tag{25}$$

Additional edges with operators for example graph



All possible directed edges to be added in the graph. Here, we add three edges in red with the corresponding operator for a simplified pictorial description.

State of the walker after teleportation

Now applying the operators of $\mathcal{D}$ on $\rho^{(t)}$ we get

$$\begin{aligned}\rho_D^{(t+1)} = \Psi_D(\rho^{(t)}) &= \sum_{u \in V(G)} \sum_{v \in V(G)} D_{u,v} \rho^{(t)} D_{u,v}^{\dagger} \\ &= \sum_{u \in V(G)} \sum_{v \in V(G)} \frac{1}{\sqrt{n}} (I_n \otimes S_{u,v}) \left(\sum_{v \in V(G)} \rho_v^{(t)} \otimes |v\rangle \langle v| \right) \frac{1}{\sqrt{n}} (I_n \otimes S_{u,v})^{\dagger} \\ &= \sum_{u \in V(G)} \frac{1}{n} \sum_{v \in V(G)} \rho_v^{(t)} \otimes S_{u,v} |v\rangle \langle v| S_{u,v}^{\dagger} \\ &= \sum_{u \in V(G)} \left(\frac{1}{n} \sum_{v \in V(G)} \rho_v^{(t)} \right) \otimes |u\rangle \langle u| . \end{aligned} \tag{26}$$

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6. Comparison of ranking in different networks
7. Concluding Remarks

Discrete-time open quantum walk with restart

- The walker may follow the hyperlinks with probability α or may be teleported to any vertex with probability $(1-\alpha)$. Hence, the state of the walker is

$$\begin{aligned}\rho^{(t+1)} &= \alpha \sum_{u \in V(G)} \sum_{v \in B(u) \cup \{u\}} C_{v,u} \rho_v^{(t)} C_{v,u}^{\dagger} \otimes |u\rangle \langle u| \\ &+ (1-\alpha) \sum_{u \in V(G)} \left(\frac{1}{n} \sum_{v \in V(G)} \rho_v^{(t)} \right) \otimes |u\rangle \langle u| \\ &= \sum_{u \in V(G)} \rho_u^{(t+1)} \otimes |u\rangle \langle u|,\end{aligned} \tag{27}$$

where

$$\rho_u^{(t+1)} = \alpha \sum_{v \in B(u) \cup \{u\}} C_{v,u} \rho_v^{(t)} C_{v,u}^{\dagger} + \frac{(1-\alpha)}{n} \sum_{v \in V(G)} \rho_v^{(t)}. \tag{28}$$

Quantum PageRank

- The probability of getting the walker at vertex u at time $(t+1)$ is given by

$$p_u^{(t+1)} = \text{trace}\, \rho_u^{(t+1)}. \tag{29}$$

Definition

We define the qPageRank vector at time t for a graph G as $q\Pi^{(t)} = (p_1^{(t)}, p_2^{(t)}, \dots, p_n^{(t)})$. The qPageRank vector of a graph G is

$$q\Pi = \lim_{t\to\infty} q\Pi^{(t)} = \lim_{t\to\infty} (p_1^{(t)}, p_2^{(t)}, \dots, p_n^{(t)}), \tag{30}$$

where $p_u^{(t)}$ is the probability of getting the walker at vertex u at time t determined by equation (29).

Procedure for calculating qPageRank

Follow the below steps to calculate the qPageRank of the vertices in a given graph:

1. **Preparation of initial state:** The initial state $\rho^{(0)}$ of the system to be prepared following equation (17). Classically, we make a list $\mathcal{L}$ of length n with the matrices $\frac{I_n}{n^2}$ corresponding to the vertices.
2. **Construction of operators:** Corresponding to every directed edge in the graph, we construct the coin operators following equation (12).
3. **Quantum evolution:** We evaluate the quantum state iteratively following equation (27). Classically, to reduce the size of our calculations, we apply equation (28) to update elements in the list $\mathcal{L}$.
4. **qPageRank measurement:** We calculate the qPageRank of a vertex by measuring the state following the equation (29). Classically, we calculate the trace of the matrices in $\mathcal{L}$ at different time steps to get $q\Pi^{(t)}$ mentioned in Definition 2.

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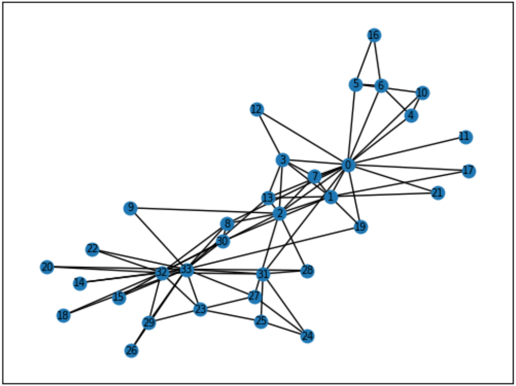
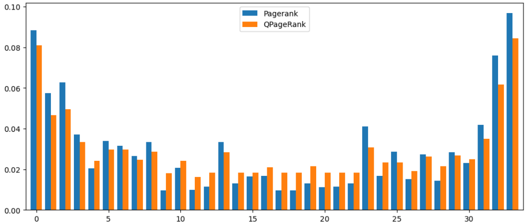
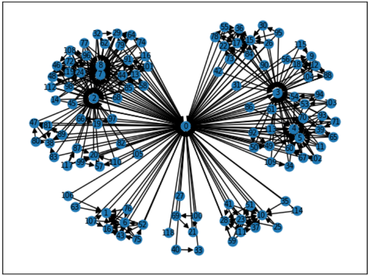
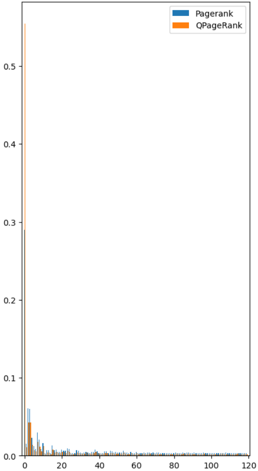
Karate club graph





Directed GNC graph





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A few previous works in this direction

1. Giuseppe Davide Paparo and Miguel Angel Martin-Delgado. "Google in a quantum network". In: *Scientific Reports* 2.1 (2012), p. 444
2. Paola Boito and Roberto Grena. "Quantum hub and authority centrality measures for directed networks based on continuous-time quantum walks". In: *Journal of Complex Networks* 9.6 (2021), cnab038
3. Wang et al., "Continuous-time quantum walk based centrality testing on weighted graphs"
4. Loke et al., "Comparing classical and quantum PageRanks"
5. Prateek Chawla, Roopesh Mangal, and C Madaiah Chandrashekar. "Discrete-time quantum walk algorithm for ranking nodes on a network". In: *Quantum Information Processing* 19.5 (2020), p. 158

and many others... For a complete list of references find [9]

[9] Supriyo Dutta. "Discrete-time open quantum walks for vertex ranking in graphs". In: *Physical Review E* 111.3 (2025), p. 034312.

Article under preparation: Community detection using quantum walk

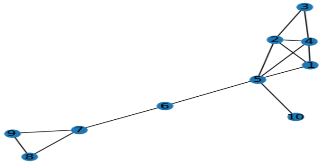

Figure: Graph for detecting communities

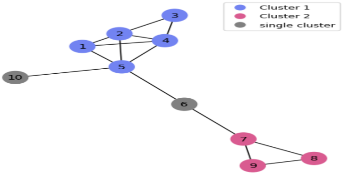

Figure: Graph with community detected by quantum walks.

For detailed proofs we refer our work

- Supriyo Dutta. "Discrete-time open quantum walks for vertex ranking in graphs". In: *Physical Review E* 111.3 (2025), p. 034312
- The article is available in ArXiv https://arxiv.org/abs/2005.03272v2

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For other works of the author

https://sites.google.com/view/supriyo-dutta/research

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